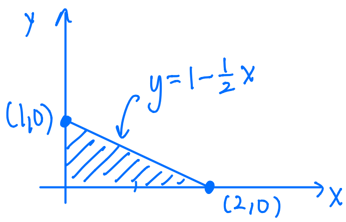
Problem 1 Center of mass in two dimensions.

(a) Find the mass and center of mass of a triangular lamina with vertices $(0,0),(2,0)$, and $(0,1)$ if the density function is $\delta(x,y)=1+x+3y$.



$$M=\iint_D \delta(x,y)\,dA=\int_0^2\int_0^{1-\frac12 x}(1+x+3y)\,dy\,dx=\int_0^2\left((1+x)y+\tfrac32 y^2\right)\Big|_{y=0}^{1-\frac12 x}dx$$

$$=\int_0^2\left((1+x)(1-\tfrac12 x)+\tfrac32(1-\tfrac12 x)^2\right)dx=\int_0^2\left(1-\tfrac12 x+x-\tfrac12 x^2+\tfrac32\left(1-x+\tfrac14 x^2\right)\right)dx$$

$$=\int_0^2\left(\tfrac52-x-\tfrac18 x^2\right)dx=\tfrac52 x-\tfrac12 x^2-\tfrac1{24}x^3\Big|_0^2=5-2-\tfrac13=\tfrac83 \qquad \boxed{\text{mass}=\tfrac83}$$

$$x_{CM}=\frac1M\iint_D x\delta(x,y)\,dA=\frac38\int_0^2\int_0^{1-\frac12 x}(x+x^2+3xy)\,dy\,dx=\frac38\int_0^2\left((x+x^2)y+\tfrac32 xy^2\right)\Big|_{y=0}^{1-\frac12 x}dx=\frac38\int_0^2\left((x+x^2)(1-\tfrac12 x)+\tfrac32 x(1-\tfrac12 x)^2\right)dx$$

$$=\frac38\int_0^2\left(x-\tfrac12 x^2+x^2-\tfrac12 x^3+\tfrac32 x\left(1-x+\tfrac14 x^2\right)\right)dx=\frac38\int_0^2\left(\tfrac52 x-x^2-\tfrac18 x^3\right)dx=\frac38\left(\tfrac54 x^2-\tfrac13 x^3-\tfrac1{32}x^4\right)\Big|_0^2=\frac38\left(5-\tfrac83-\tfrac12\right)$$

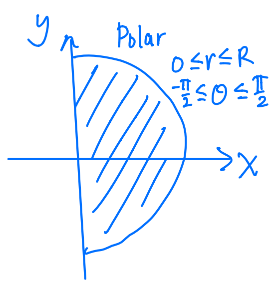
$$=\frac38\left(\tfrac{30}{6}-\tfrac{16}{6}-\tfrac36\right)=\frac38\left(\tfrac{11}{6}\right)=\frac{11}{16}$$

$$y_{CM}=\frac1M\iint_D y\,\delta(x,y)\,dA=\frac38\int_0^2\int_0^{1-\frac12 x}\left((1+x)y+3y^2\right)dy\,dx=\frac38\int_0^2\left((1+x)\tfrac{y^2}{2}+y^3\right)\Big|_{y=0}^{1-\frac12 x}dx=\frac38\int_0^2\left((1+x)\frac{(1-\frac12 x)^2}{2}+(1-\tfrac12 x)^3\right)dx$$

$$=\frac38\int_0^2\left(\tfrac12(1+x)\left(1-x+\tfrac14 x^2\right)+(1-\tfrac12 x)\left(1-x+\tfrac14 x^2\right)\right)dx=\frac38\int_0^2\left(\tfrac12-\cancel{\tfrac12 x}+\tfrac18 x^2+\cancel{\tfrac12 x}-\cancel{\tfrac12 x^2}+\cancel{\tfrac18 x^3}+1-x+\tfrac14 x^2-\tfrac12 x+\cancel{\tfrac12 x^2}-\cancel{\tfrac18 x^3}\right)dx$$

$$=\frac38\int_0^2\left(\tfrac32-\tfrac32 x+\tfrac38 x^2\right)dx=\frac38\left(\tfrac32 x-\tfrac34 x^2+\tfrac18 x^3\right)\Big|_0^2=\frac38(3-3+1)=\frac38 \qquad \boxed{\text{Center of mass: }\left(\tfrac{11}{16},\tfrac38\right)}$$

(b) Find the mass and center of mass of a semicircular lamina $x^2+y^2\le R^2, x\ge 0$ if the density function is $\delta(x,y)=C\sqrt{x^2+y^2}$ for some constant C.



$$M=\iint_D \delta(x,y)\,dA=\int_{-\frac\pi2}^{\frac\pi2}\int_0^R \underbrace{Cr\cdot r}_{r^2}\,dr\,d\theta=C\int_{-\frac\pi2}^{\frac\pi2}\frac{r^3}{3}\Big|_{r=0}^R d\theta=\frac{CR^3}{3}\int_{-\frac\pi2}^{\frac\pi2}d\theta=\frac{CR^3}{3}\theta\Big|_{-\frac\pi2}^{\frac\pi2}$$

$$=\frac{CR^3}{3}\pi \qquad \boxed{\text{mass: }\frac{CR^3\pi}{3}}$$

$$x_{CM}=\frac1M\iint_D x\delta(x,y)\,dA=\frac{3}{\cancel{C}R^3\pi}\int_{-\frac\pi2}^{\frac\pi2}\int_0^R r\cos\theta\cdot\cancel{C}r\cdot r\,dr\,d\theta=\frac{3}{R^3\pi}\int_{-\frac\pi2}^{\frac\pi2}\int_0^R r^3\cos\theta\,dr\,d\theta$$

$$=\frac{3}{R^3\pi}\int_{-\pi/2}^{\pi/2}\frac{r^4}{4}\Big|_{r=0}^R\cos\theta\,d\theta=\frac{3}{\cancel{R^3}\pi}\cdot\frac{\cancel{R^4}\,R}{4}\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta=\frac{3R}{4\pi}(\sin\theta)\Big|_{-\pi/2}^{\pi/2}=\frac{3R}{4\pi}(1-(-1))=\frac{3R}{2\pi}$$

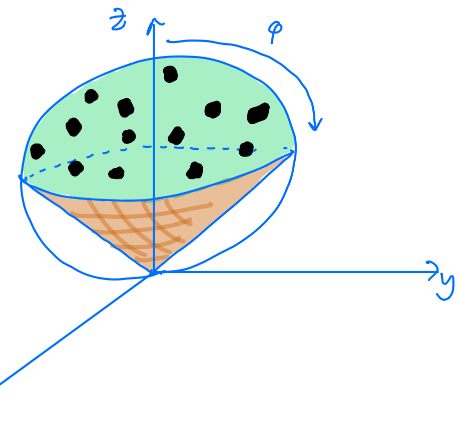
$$y_{CM}=\frac1M\iint_D y\,\delta(x,y)\,dA=\frac{3}{\cancel{C}R^3\pi}\int_{-\pi/2}^{\pi/2}\int_0^R r\sin\theta\cdot\cancel{C}r\cdot r\,dr\,d\theta=\frac{3}{R^3\pi}\int_{-\pi/2}^{\pi/2}\frac{r^4}{4}\Big|_{r=0}^R\sin\theta\,d\theta=\frac{3}{\cancel{R^3}\pi}\frac{\cancel{R^4}\,R}{4}\int_{-\pi/2}^{\pi/2}\sin\theta\,d\theta$$

$$=\frac{3R}{4\pi}(-\cos\theta)\Big|_{\theta=-\pi/2}^{\pi/2}=\frac{3R}{4\pi}(0-0)=0 \qquad \boxed{\text{Center of mass: }\left(\tfrac{3R}{2\pi},0\right)}$$

Problem 2 Center of mass in three dimensions. Suppose you are eating an ice cream cone. You determine the ice cream cone can be modeled as a solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$. If the density of the ice cream cone has uniform density $\cancel{\delta(x, y)}$ $\delta(x,y,z)$ $= 1$, determine the mass and center of mass of the ice cream cone.

* Rewrite $x^2+y^2+z^2=z$ in "standard form" to sketch ice cream cone

$$x^2+y^2+z^2-z=0$$
$$x^2+y^2+z^2-z+\tfrac{1}{4}=\tfrac{1}{4}$$
$$x^2+y^2+\left(z-\tfrac{1}{2}\right)^2=\left(\tfrac{1}{2}\right)^2 \qquad \text{center } (0,0,\tfrac{1}{2}), \text{ radius } \tfrac{1}{2}$$



Use spherical coordinates

- $z=\sqrt{x^2+y^2} \Rightarrow \rho\cos\varphi = \sqrt{\rho^2\cos^2\theta\sin^2\varphi+\rho^2\sin^2\theta\sin^2\varphi} = \sqrt{\rho^2\sin^2\varphi(\cos^2\theta+\sin^2\theta)} = \rho\sin\varphi \Rightarrow \tan\varphi=1$, $\varphi=\frac{\pi}{4}$ i.e. $0\le\varphi\le\pi/4$

- $x^2+y^2+z^2=z \Rightarrow \rho^2=\rho\cos\varphi \Rightarrow \rho=\cos\varphi$ i.e. $0\le\rho\le\cos\varphi$

$$M=\iiint_W \delta(x,y,z)\,dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\varphi} 1\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \int_0^{2\pi}\int_0^{\pi/4}\frac{\rho^3}{3}\bigg|_{\rho=0}^{\cos\varphi}\sin\varphi\,d\varphi\,d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4}\cos^3\varphi\sin\varphi\,d\varphi\,d\theta$$

$u=\cos\varphi \quad du=-\sin\varphi\,d\varphi$

$$=-\frac{1}{3}\int_0^{2\pi}\int_1^{1/\sqrt{2}}u^3\,du\,d\theta = -\frac{1}{3}\int_0^{2\pi}\frac{u^4}{4}\bigg|_{u=1}^{1/\sqrt{2}}d\theta = \frac{-1}{12}\left(\left(\frac{1}{\sqrt{2}}\right)^4-1^4\right)\theta\bigg|_{\theta=0}^{2\pi} = \frac{-1}{12}\cdot\left(\frac{-3}{4}\right)\cdot 2\pi = \frac{6\pi}{48}=\frac{\pi}{8}$$

$\boxed{\text{mass: } \frac{\pi}{8}}$

* for center of mass we expect $x_{CM}=y_{CM}=0$.

$$x_{CM}=\frac{1}{M}\iiint_W x\,\delta(x,y,z)\,dV = \frac{8}{\pi}\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\varphi}\rho\cos\theta\sin\varphi\cdot 1\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta \underset{\substack{\uparrow\\ \text{switch order}}}{=} \frac{8}{\pi}\int_0^{\pi/4}\int_0^{\cos\varphi}\int_0^{2\pi}\rho^3\sin^2\varphi\cos\theta\,d\theta\,d\rho\,d\varphi$$

$$=\frac{8}{\pi}\int_0^{\pi/4}\int_0^{\cos\varphi}\sin\theta\bigg|_{\theta=0}^{2\pi}\rho^3\sin^2\varphi\,d\rho\,d\varphi = 0.$$

$$y_{CM}=\frac{1}{M}\iiint_W y\,\delta(x,y,z)\,dV = \frac{8}{\pi}\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\varphi}\rho\sin\theta\sin\varphi\cdot 1\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta \underset{\substack{\uparrow\\ \text{switch order}}}{=} \frac{8}{\pi}\int_0^{\pi/4}\int_0^{\cos\varphi}\int_0^{2\pi}\rho^3\sin^2\varphi\sin\theta\,d\theta\,d\rho\,d\varphi$$

$$=\frac{8}{\pi}\int_0^{\pi/4}\int_0^{\cos\varphi}-\cos\theta\bigg|_{\theta=0}^{2\pi}\rho^3\sin^2\varphi\,d\rho\,d\varphi = 0.$$

$$z_{CM}=\frac{1}{M}\iiint_W z\,\delta(x,y,z)\,dV = \frac{8}{\pi}\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\varphi}\rho\cos\varphi\cdot 1\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \frac{8}{\pi}\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\varphi}\rho^3\sin\varphi\cos\varphi\,d\rho\,d\varphi\,d\theta$$

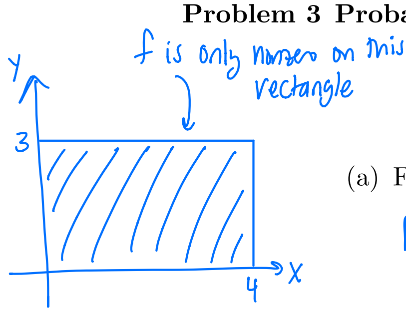
$$=\frac{\cancel{8}^{\,2}}{\pi}\int_0^{2\pi}\int_0^{\pi/4}\frac{\rho^4}{\cancel{4}}\bigg|_{\rho=0}^{\cos\varphi}\sin\varphi\cos\varphi\,d\varphi\,d\theta = \frac{2}{\pi}\int_0^{2\pi}\int_0^{\pi/4}\cos^5\varphi\sin\varphi\,d\varphi\,d\theta = -\frac{2}{\pi}\int_0^{2\pi}\int_1^{1/\sqrt{2}}u^5\,du\,d\theta$$

$u=\cos\varphi \quad du=-\sin\varphi\,d\varphi$

$$=-\frac{\cancel{2}}{\pi}\int_0^{2\pi}\frac{u^6}{\cancel{6}_{\,3}}\bigg|_{u=1}^{1/\sqrt{2}}d\theta = \frac{-1}{3\pi}\int_0^{2\pi}\left[\left(\frac{1}{\sqrt{2}}\right)^6-1\right]d\theta = \frac{-1}{3\pi}\left(\frac{1}{8}-1\right)\theta\bigg|_0^{2\pi} = \frac{-1}{3\pi}\cdot\frac{-7}{8}\cdot 2\pi = \frac{14\pi}{24\pi}=\frac{7}{12}$$

$\boxed{\text{center of mass: } \left(0,0,\frac{7}{12}\right)}$

Problem 3 Probability. The joint density function for a random variables X and Y is



$$f(x,y)=\begin{cases} C(x+y) & \text{if } 0\le x\le 4, 0\le y\le 3\\ 0 & \text{otherwise.}\end{cases}$$

(a) Find the value of the constant C.

Need $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dA = 1.$

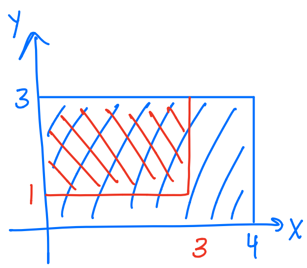
But $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dA = \int_0^4\int_0^3 C(x+y)\,dy\,dx = C\int_0^4 \left(xy+\frac{y^2}{2}\right)\Big|_{y=0}^{3} dx$

$= C\int_0^4 \left(3x+\frac{9}{2}\right)dx = C\left(\frac{3x^2}{2}+\frac{9}{2}x\right)\Big|_{x=0}^{4} = C\left(\frac{3\cdot 16}{2}+\frac{9}{2}\cdot 4\right)$

$= C(24+18) = 42C$

$\Rightarrow \quad 42C=1 \quad \Rightarrow \quad C=\frac{1}{42}$

(b) Find $P(X\le 3, Y\ge 1)$.

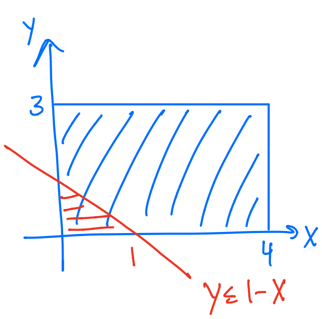


$P(X\le 3, Y\ge 1) = \int_0^3\int_1^3 \frac{1}{42}(x+y)\,dy\,dx = \frac{1}{42}\int_0^3 \left(xy+\frac{y^2}{2}\right)\Big|_{y=1}^{3} dx$

$= \frac{1}{42}\int_0^3 \left(3x+\frac{9}{2}-x-\frac{1}{2}\right)dx = \frac{1}{42}\int_0^3 (2x+4)\,dx = \frac{1}{42}(x^2+4x)\Big|_{x=0}^{3}$

$= \frac{1}{42}(9+12) = \frac{21}{42} = \frac{1}{2}$

(c) Find $P(X+Y\le 1)$.



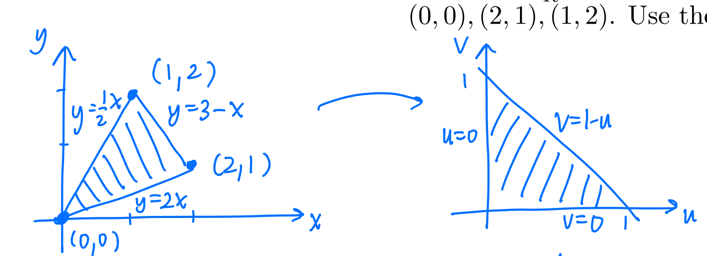
$P(X+Y\le 1) = \int_0^1\int_0^{1-x} \frac{1}{42}(x+y)\,dy\,dx = \frac{1}{42}\int_0^1 \left(xy+\frac{y^2}{2}\right)\Big|_{y=0}^{1-x} dx$

$= \frac{1}{42}\int_0^1 \left(x(1-x)+\frac{(1-x)^2}{2}\right)dx = \frac{1}{42}\int_0^1 \left(\cancel{x}-x^2+\frac{1}{2}-\cancel{x}+\frac{1}{2}x^2\right)dx$

$= \frac{1}{42}\int_0^1 \left(\frac{1}{2}-\frac{1}{2}x^2\right)dx = \frac{1}{42}\left(\frac{1}{2}x-\frac{1}{6}x^3\right)\Big|_{x=0}^{1} = \frac{1}{42}\left(\frac{1}{2}-\frac{1}{6}\right) = \frac{1}{42}\cdot\frac{1}{3} = \frac{1}{126}$

Problem 4 Change of Variables.

(a) Evaluate $\iint_R (x-3y)dA$ where R is the triangular region with the vertices $(0,0),(2,1),(1,2)$. Use the transformation $x = 2u+v, y = u+2v$.



Bounds for new domain: $0 \le u \le 1$
$0 \le v \le 1-u$

Jacobian: $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4-1=3$

$y=\frac{1}{2}x \Rightarrow u+2v = \frac{1}{2}(2u+v) \Rightarrow u+2v = u+\frac{1}{2}v \Rightarrow v=0$

$y=2x \Rightarrow u+2v = 2(2u+v) \Rightarrow u+2v = 4u+2v \Rightarrow u=0$

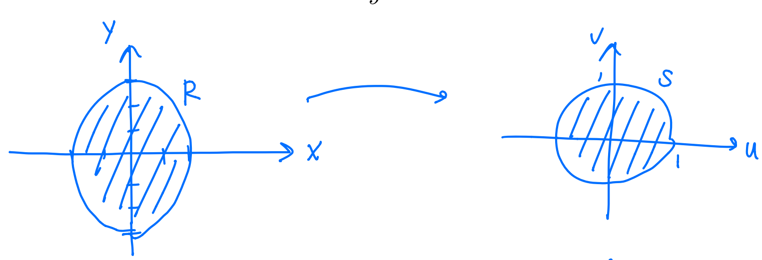
$y=3-x \Rightarrow u+2v = 3-(2u+v) \Rightarrow u+2v = 3-2u-v \Rightarrow 3v = 3-3u$
$v=1-u$

$$\iint_R (x-3y)\,dA = \int_0^1\int_0^{1-u} \left(2u+v-3(u+2v)\right)|3|\,dv\,du = 3\int_0^1\int_0^{1-u}(-u-5v)\,dv\,du = 3\int_0^1 \left(-uv-\frac{5}{2}v^2\right)\Big|_{v=0}^{1-u} du$$

$$= 3\int_0^1 \left(-u(1-u)-\frac{5}{2}(1-u)^2\right)dv = 3\int_0^1\left(-u+u^2-\frac{5}{2}(1-2u+u^2)\right)du = 3\int_0^1\left(-\frac{5}{2}+4u-\frac{3}{2}u^2\right)du$$

$$= 3\left(-\frac{5}{2}u+2u^2-\frac{1}{2}u^3\right)\Big|_{u=0}^{1} = 3\left(-\frac{5}{2}+2-\frac{1}{2}\right) = 3(2-3) = -3$$

(b) Evaluate $\iint_R x^2 dA$ where R is the region bounded by the ellipse $9x^2 + 4y^2 = 36$. Use the transformation $x = 2u, y = 3v$.



Jacobian: $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$

$9x^2+4y^2=36 \Rightarrow 9(2u)^2+4(3u^2)=36 \Rightarrow u^2+v^2=1$

$$\iint_R x^2\,dA = \iint_S (2u)^2|6|\,dA = 24\iint_S u^2\,dA = 24\int_0^{2\pi}\int_0^1 r^2\cos^2\theta\, r\,dr\,d\theta = 24\int_0^{2\pi}\int_0^1 r^3\cos^2\theta\,dr\,d\theta$$

use polar

$$= 24\int_0^{2\pi}\frac{r^4}{4}\Big|_{r=0}^{1}\cos^2\theta\,d\theta = 6\int_0^{2\pi}\frac{1}{2}(1+\cos 2\theta)\,d\theta = 3\left(\theta+\frac{\sin 2\theta}{2}\right)\Big|_{\theta=0}^{2\pi} = 6\pi$$