Problem 1 Ben sits his first midterm for Math 32B and gets the following question:

Let W be the region above $z = x^2 + y^2$ and below $z = 5$ and bounded by $y = 0$ and $y = 1$. Calculate $\int_W y dV$.

(a) After working on it for a while, he eventually arrives at the answer of $-\frac{\sqrt{5}}{5}$. Does this seem reasonable? Why/Why not?

Solution. The function we are integrating is positive over the region we are integrating but we got a negative answer which is impossible.

(b) Here is Ben's working out:

$$= \int_{-\sqrt{5}}^{\sqrt{5}}\int_0^1\int_{x^2+y^2}^5 y \, dz \, dy \, dx = \int_{-\sqrt{5}}^{\sqrt{5}}\int_0^1 \frac{y^2}{2}\Big|_{x^2+y^2}^5 dy\,dx$$

$$= \int_{-\sqrt{5}}^{\sqrt{5}}\int_0^1 \frac{25}{2} - \frac{(x^2+y^2)^2}{2} dy\,dx$$

$$= \int\int \frac{25}{2} dy\,dx - \int\int \frac{(x^2+y^2)^2}{2} dy\,dx$$

$$= 25\sqrt{5} - \frac{1}{2}\int x^4 + y^4 dy\,dx$$

$$= 25\sqrt{5} - \frac{1}{2}\int_{-\sqrt{5}}^{\sqrt{5}}\int_0^1 x^4 dy\,dx - \frac{1}{2}\int_{-\sqrt{5}}^{\sqrt{5}}\int_0^1 y^4 dy\,dx$$

$$= 25\sqrt{5} - \frac{1}{2} x^5\Big|_{-\sqrt{5}}^{\sqrt{5}} - \sqrt{5}\cdot\frac{1}{5}$$

$$= 25\sqrt{5} - 25\sqrt{5} - \frac{1}{5}\sqrt{5} = -\frac{1}{5}\sqrt{5}.$$

Can you find all the mistakes that Ben made?

Solution. Here are some things that Ben did wrong:

1. The bounds are wrong.
2. He integrated the y when he should have integrated with respect to the z.
3. He expanded $(x^2 + y^2)^2$ incorrectly.
4. Forgot a $\frac{1}{5}$ factor when integrating x^4.

There are also some minor things, like didn't write bounds all the time or dropped the number of required integral signs, but whether this is a mistake or not is up to personal taste.

(c) Did you find Ben's answer hard or easy to read? Come up with things that Ben did right in his answer that made it easy to read and things that Ben could do to help make his answer easier to read.

Solution. There are no right or wrong answers here and it can depend on personal preference. Here are some things I would say:

Good things:

- The answer is written in order from top to bottom. No jumping around the page.

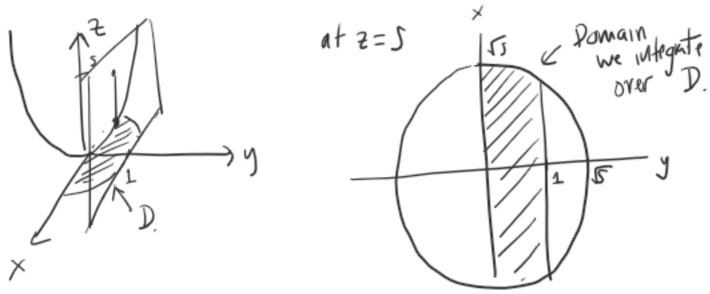
Things that can improve this answer:

- Pictures of the domain/thing we are integrating over.

- Connecting words and sentences that explain the thought process being done.

(d) Rewrite Ben's answer correctly and compare your answer with other people in your group. Don't just compare the end result but read each other's solutions and see what they did/didn't do that made it easy to read.

Solution. Sample solution: (I'm not very good at pictures)



over a point in D, the z-values are from x^2+y^2 to 5.

Hence: $\iiint_W y\,dV = \iint_D \int_{x^2+y^2}^{5} y\,dz\,dA$

$$= \iint_D 5y - y(x^2+y^2)\,dA$$

$$= \iint_D y(5-y^2) - yx^2\,dA.$$

Now, D is given by bounds $0 \le y \le 1$, $-\sqrt{5-y^2} \le x \le \sqrt{5-y^2}$

so $\iint_D y(5-y^2) - yx^2\,dA = \int_0^1 \int_{-\sqrt{5-y^2}}^{\sqrt{5-y^2}} y(5-y^2) - yx^2\,dx\,dy$

$$= 2\int_0^1 yx(5-y^2) - y\frac{x^3}{3}\Big|_0^{\sqrt{5-y^2}} dy \quad \text{by symmetry}$$

$$= 2\int_0^1 y(5-y^2)^{3/2} - y\frac{(5-y^2)^{3/2}}{3}\,dy$$

$$= \frac{4}{3}\int_0^1 y(5-y^2)^{3/2}\,dy$$

$$= \frac{4}{3}\left[-\frac{1}{2}\cdot\frac{2}{5}(5-y^2)^{5/2}\right]_0^1$$

$$= \frac{-4}{15}\left(4^{5/2} - 5^{5/2}\right)$$

$$= \frac{4}{15}\left(5^{5/2} - 4^{5/2}\right)$$

Your answer could look very different to this. The ultimate goal of this question is to get you to think about how you set out and style your answers. Effectively communicating your ideas is an important aspect of mathematics that isn't always talked about.

Problem 2 We will integrate $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ in this question using a trick with polar coordinates.

(a) Let $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$. Justify why $I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dxdy$.

Solution. The variable inside the integral can be changed without affecting anything. So we get

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-x^2}\,dx \\ &= \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dxdy. \end{aligned}$$

(b) Don't question your TA when they say that

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dxdy = \lim_{R\to\infty} \iint_{B_R} e^{-x^2-y^2}\,dxdy$$

where B_R is the ball centered at zero of radius R.

(c) Use polar coordinates to calculate $\iint_{B_R} e^{-x^2-y^2}\,dxdy$.

Solution.

$$\begin{aligned} \iint_{B_R} e^{-x^2-y^2}\,dxdy &= \int_0^{2\pi}\int_0^R re^{-r^2}\,drd\theta \\ &= 2\pi \int_0^R re^{-r^2}\,dr \\ &= -\pi e^{-r^2}|_0^R \\ &= \pi(1-e^{-R^2}) \end{aligned}$$

(d) What is I equal to?

Solution. We take the limit as $R \to \infty$ and get that $I^2 = \pi$ and so $I = \sqrt{\pi}$.

Problem 3 It is always a good idea to keep an eye out for symmetries in double and triple integrals in order to reduce calculations needed. Without calculating anything, what are the following integrals?

(a) $\int_{-1}^{1}\int_{-2}^{2}\int_{0}^{4-x^2} x^2z^3\ln(x^2+y^2-z^2+2)\,dydxdz$

Solution. Notice that the function is odd in z and the domain of integration is symmetric around the xy-plane. Hence all the values cancel and we get that this integral is equal to zero.

(b) $\int_{-1}^{1}\int_{-1}^{1} 2 + y^2x^3 - x^2\sin(y)dxdy$

Solution. The function y^2x^3 is odd in x and $x^2\sin(y)$ is odd in y. Hence these parts will go to zero over the domain of integration. So we are just left with 2 over a domain of area 4. Hence the integral is 8.

(c) $\int_{1}^{2}\int_{1}^{2} \frac{x^2}{x^2+y^2}dxdy$

Solution. As stated previously, changing the letters used inside integrals doesn't change anything (as long as different letters stay different). Hence, if we swap x and y, this won't change the integral. Hence,

$$\begin{aligned}\int_{1}^{2}\int_{1}^{2} \frac{x^2}{x^2+y^2}dxdy &= \frac{1}{2}\left(\int_{1}^{2}\int_{1}^{2} \frac{x^2}{x^2+y^2}dxdy + \int_{1}^{2}\int_{1}^{2} \frac{y^2}{x^2+y^2}dydx\right)\\ &= \frac{1}{2}\int_{1}^{2}\int_{1}^{2} \frac{x^2+y^2}{x^2+y^2}dxdy\\ &= \frac{1}{2}\end{aligned}$$

Problem 4 A riddle: I am a two-variable function. I can tell how far a given point is from the origin but not how far from either the x or y-axis. My average value over any annulus is always equal to the reciprocal of the sum of the inner and outer radius. What function am I?

Solution. Let $f(x,y)$ be the two variable function. Since it can tell how far a point is from the origin but not any of the axis, it must only depend on the quantity $r = \sqrt{x^2+y^2}$. Hence, $f(x,y) = g(r)$ for some one variable function g. The condition on the average value means that for any annulus with inner radius ρ and outer radius R we have

$$\begin{aligned}\frac{2\pi\int_{\rho}^{R} rg(r)dr}{\pi(R^2-\rho^2)} &= \frac{1}{R+\rho}\\ \int_{\rho}^{R} rg(r)dr &= \frac{R-\rho}{2}\end{aligned}$$

This must be true for any R, ρ and so we must have $rg(r) = \frac{1}{2}$ and so $f(x,y) = \frac{1}{2\sqrt{x^2+y^2}}$

Problem 5 It's time for the annual triple integral beauty pageant. After hearing about the amazing prize (admiration from your TA and fellow students), you decide to enter it. Can you come up with an interesting triple integral that evaluates to exactly 2020?