Problem 1 Double integrals over rectangles. Evaluate the following integrals:

(a) $\iint_R (x+y)dxdy$ where $R = \{(x,y) : 0 \le x \le 3, 0 \le y \le 1\}$

$$\int_0^1\int_0^3 (x+y)\,dxdy = \int_0^1 \left(\tfrac{1}{2}x^2 + xy\right)\Big|_{x=0}^{3} dy = \int_0^1 \left(\tfrac{9}{2} + 3y\right)dy = \tfrac{9}{2}y + \tfrac{3}{2}y^2\Big|_{y=0}^{1}$$

$$= \tfrac{9}{2} + \tfrac{3}{2} = \tfrac{12}{2} = 6$$

(b) $\int_0^1 \int_1^2 \frac{xe^x}{y} dydx$ (*Hint:* use integration by parts, $\int udv = uv - \int vdu$)

$$\int_0^1\int_1^2 \frac{xe^x}{y}\,dy\,dx = \int_0^1 xe^x \ln y\Big|_{y=1}^{2} dx = \int_0^1 xe^x (\ln 2 - \cancel{\ln 1}^{0})\,dx$$

$$= \ln 2 \int_0^1 xe^x\,dx \qquad = \ln 2\left(xe^x\Big|_{x=0}^{1} - \int_0^1 e^x\,dx\right)$$

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$$= \ln 2\left(xe^x - e^x\right)\Big|_{x=0}^{1} = \ln 2\left(1\cancel{e} - \cancel{e} - (0e^0 - e^0)\right)$$

$$= \ln 2$$

(c) $\int_0^1 \int_0^\pi r\sin^2\theta d\theta dr$ (*Hint:* use $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$)

$$\int_0^1\int_0^\pi r\sin^2\theta\,d\theta\,dr = \int_0^1\int_0^\pi r\cdot\tfrac{1}{2}(1-\cos 2\theta)\,d\theta\,dr = \tfrac{1}{2}\int_0^1 r\left(\theta - \tfrac{\sin 2\theta}{2}\right)\Big|_{\theta=0}^{\pi} dr$$

$$= \tfrac{1}{2}\int_0^1 \pi r\,dr = \tfrac{\pi}{2}\cdot\tfrac{1}{2}r^2\Big|_0^1 = \tfrac{\pi}{4}$$

Problem 2 Double integrals over general regions. Evaluate the following integrals:

(a) $\int_0^2 \int_x^{2x} xy dy dx$

$$\int_0^2 \int_x^{2x} xy\, dy\, dx = \int_0^2 x \frac{y^2}{2}\Big|_{y=x}^{2x} dx = \int_0^2 x\cdot\left(\frac{(2x)^2}{2} - \frac{x^2}{2}\right) dx$$

$$= \int_0^2 x\left(\frac{4x^2}{2} - \frac{x^2}{2}\right) dx = \int_0^2 \frac{3x^3}{2} dx = \frac{3}{2}\cdot\frac{x^4}{4}\Big|_{x=0}^{2} = \frac{3}{2}\cdot\left(\frac{16}{4} - \frac{0}{4}\right)$$

$$= \frac{3}{2}\cdot 4 = 6$$

(b) $\iint_D 2x\sqrt{y^2 - x^2} dx dy$ where $D = \{(x, y) : 0 \le y \le 1, 0 \le x \le y\}$

$$\int_0^1 \int_0^y 2x\sqrt{y^2-x^2}\, dx\, dy = -\int_0^1 \int_{y^2}^0 \sqrt{u}\, du\, dy = \int_0^1 \int_0^{y^2} u^{1/2}\, du\, dy$$

$u = y^2 - x^2$ when $x = 0, u = y^2$

$du = -2x dx$ $\quad x = y, u = 0$

$$= \int_0^1 \frac{2}{3} u^{3/2}\Big|_{u=0}^{y^2} dy = \int_0^1 \frac{2}{3} y^3 dy = \frac{2}{3}\cdot\frac{y^4}{4}\Big|_{y=0}^{1}$$

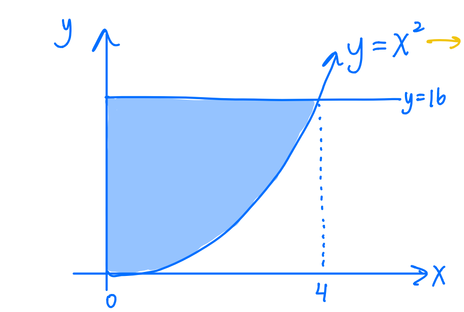
$$= \frac{2}{3}\cdot\frac{1}{4} = \frac{1}{6}$$

Problem 3 Switching the order of integration.

(a) Sketch the region of integration and change the order of integration:

$$\int_0^4 \int_{x^2}^{16} f(x, y) dy dx = \int_0^{16} \int_0^{\sqrt{y}} f(x, y) dx dy$$

$0 \le x \le 4$
$x^2 \le y \le 16$



$0 \le y \le 16$

$0 \le x \le \sqrt{y}$

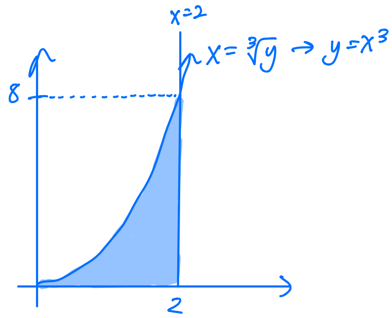
left bounding function

right bounding function

(b) Evaluate the integral by switching the order of integration:

$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\, dx dy$$

$0 \le y \le 8$
$\sqrt[3]{y} \le x \le 2$



$0 \le x \le 2$
$0 \le y \le x^3$

$\int_0^2 \int_0^{x^3} e^{x^4}\, dy\, dx = \int_0^2 y e^{x^4} \Big|_{y=0}^{x^3} dx = \int_0^2 x^3 e^{x^4} dx$

$u = x^4$
$du = 4x^3 dx$

$= \frac{1}{4}\int_0^{16} e^u du = \frac{1}{4} e^u \Big|_0^{16} = \boxed{\frac{1}{4}(e^{16}-1)}$

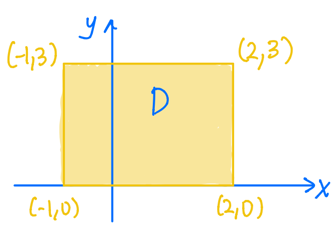
when $x=0$, $u=0$
$x=2$ $u=16$

Problem 4 **Average value.** The average value of a function $f(x,y)$ over a region D is given by

$$f_{ave} = \frac{1}{A}\iint_D f(x,y)dA$$

where $A = \iint_D 1dA$, i.e. the area of the region D. Use this formula to find the average value of the given function over the given domain.

(a) $f(x,y) = xy^2$, the rectangle with vertices $(-1,0), (-1,3), (2,3), (2,0)$



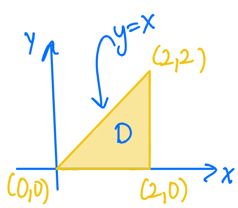
$-1 \le x \le 2$
$0 \le y \le 3$

$A = (2-(-1))(3-0) = 3(3) = 9$

$f_{ave} = \frac{1}{9}\int_{-1}^2 \int_0^3 xy^2\, dy\, dx = \frac{1}{9}\int_{-1}^2 \frac{xy^3}{3}\Big|_{y=0}^3 dx = \frac{1}{9}\int_{-1}^2 \left(\frac{27x}{3}\right) dx$

$= \int_{-1}^2 x\, dx = \frac{x^2}{2}\Big|_{x=-1}^2 = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$

(b) $f(x,y) = x - y$, the triangle with vertices $(0,0), (2,0), (2,2)$



$0 \le x \le 2$
$0 \le y \le x$

$A = \frac{1}{2}(2)(2) = 2$

$f_{ave} = \frac{1}{2}\int_0^2 \int_0^x (x-y)\, dy\, dx = \frac{1}{2}\int_0^2 \left(xy - \frac{y^2}{2}\right)\Big|_{y=0}^x dx$

$= \frac{1}{2}\int_0^2 \left(x^2 - \frac{x^2}{2}\right) dx = \frac{1}{4}\int_0^2 x^2 dx = \frac{1}{4}\frac{x^3}{3}\Big|_{x=0}^2 = \frac{8}{12} = \frac{2}{3}$

Problem 5 Triple Integrals. Evaluate the following integrals:

(a) $\int_0^1 \int_0^1 \int_0^1 (9 - xz^2)dxdydz$

$$\int_0^1\int_0^1\int_0^1 (9-xz^2)\,dx\,dy\,dz = \int_0^1\int_0^1 \left(9x - \frac{x^2z^2}{2}\right)\Big|_{x=0}^{1} dy\,dz$$

$$= \int_0^1\int_0^1 \left(9-\frac{z^2}{2}\right)dy\,dz = \int_0^1 \left(9-\frac{z^2}{2}\right)y\Big|_{y=0}^{1} dz = \int_0^1 \left(9-\frac{z^2}{2}\right)dz$$

$$= 9z - \frac{z^3}{6}\Big|_{z=0}^{1} = 9 - \frac{1}{6} = \frac{53}{6}$$

(b) $\int_0^2 \int_0^z \int_0^y 3ze^{-y^2}dxdydz$

$$\int_0^2\int_0^z\int_0^y 3ze^{-y^2}\,dx\,dy\,dz = \int_0^2\int_0^z 3ze^{-y^2}x\Big|_{x=0}^{y} dy\,dz$$

$$= \int_0^2\int_0^z 3zye^{-y^2}dy\,dz = -\frac{1}{2}\int_0^2\int_0^{-z^2} 3z e^u\,du\,dz = -\frac{1}{2}\int_0^2 3ze^u\Big|_{u=0}^{-z^2} dz$$

$u = -y^2$ when $y=0, u=0$

$du = -2y\,dy$ $y=z, u=-z^2$

$$= -\frac{1}{2}\int_0^2 3z(e^{-z^2}-1)\,dz = -\frac{1}{2}\left(\int_0^2 3ze^{-z^2}dz - \int_0^2 3z\,dz\right)$$

$w = -z^2$ when $z=0, w=0$

$dw = -2z\,dz$ $z=2, w=-4$

$$= -\frac{1}{2}\left(-\frac{1}{2}\int_{-4}^0 3e^w\,dw - \frac{3z^2}{2}\Big|_{z=0}^{2}\right) = -\frac{1}{2}\left(-\frac{1}{2}3e^w\Big|_{w=-4}^{0} - 6\right)$$

$$= \underbrace{-\frac{1}{2}\left(-\frac{1}{2}\cdot 3(1-e^{-4}) - 6\right)}_{\text{on an exam OK to stop here}} = \frac{-3}{4}(1-e^{-4}) + 3 = -\frac{3}{4} + \frac{3}{4}e^{-4} + \frac{12}{4} = \frac{3}{4}e^{-4} + \frac{9}{4}$$