* Solutions *

Thursday, January 9th, 2020 Math 32B Worksheet #1 TA: Victoria Kala

Welcome to Math 32B! This worksheet is designed to help you become comfortable dealing with and computing Riemann sums.

Problem 1 Let $\mathcal{R} = [1, 5] \times [1.5, 3]$. In this problem, our goal is to approximate the integral

$$\iint_{\mathcal{R}} (xy + y)\, \mathrm{d}A$$

using a Riemann sum.

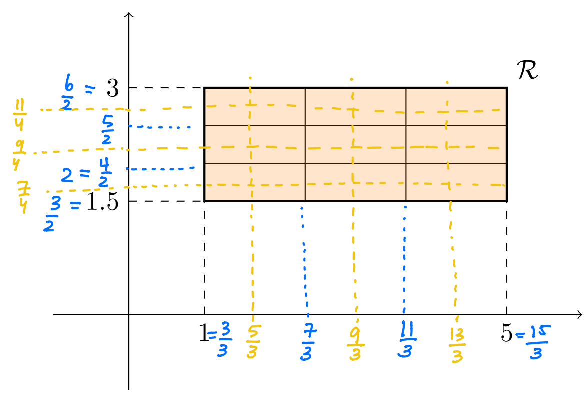
Recall: the point of Riemann sums is that they approximate integrals by breaking a domain into regions, looking at the value of the function at one point in each region, and then summing the volumes of rectangular prisms with those heights. By choosing smaller and smaller regions to break our domain into, the Riemann sums become better and better approximations of the integral (regardless of which points we pick from each region). In our case, the domain $\mathcal{R}$ is a rectangle, and while we could chop this rectangle up in many different ways, the simplest way to subdivide it is to split it evenly into smaller rectangles.

Here's the subdivision we will use:

$\frac{\frac{5}{2}+\frac{6}{2}}{2} = \frac{\frac{11}{2}}{2} = \frac{11}{4}$

$\frac{\frac{4}{2}+\frac{5}{2}}{2} = \frac{\frac{9}{2}}{2} = \frac{9}{4}$

$\frac{\frac{3}{2}+\frac{4}{2}}{2} = \frac{\frac{7}{2}}{2} = \frac{7}{4}$



$\Delta x = \frac{5-1}{3} = \frac{4}{3}$

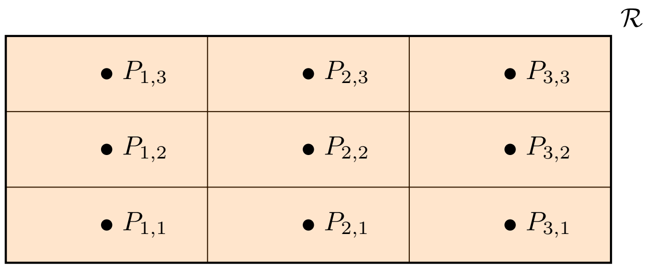
$\Delta y = \frac{3-1.5}{3} = \frac{1.5}{3} = \frac{\frac{3}{2}}{3} = \frac{1}{2}$

$\frac{\frac{3}{3}+\frac{7}{3}}{2} = \frac{\frac{10}{3}}{2} = \frac{5}{3}$ $\frac{\frac{7}{3}+\frac{11}{3}}{2} = \frac{\frac{18}{3}}{2} = \frac{9}{3}$ $\frac{\frac{11}{3}+\frac{15}{3}}{2} = \frac{\frac{26}{3}}{2} = \frac{13}{3}$

In the above picture, $\mathcal{R}$ is split evenly in thirds along both axes (resulting in 9 total small rectangles) – the Riemann sum we will end up with is often called $S_{3,3}$. To compute $S_{3,3}$, we need to go through the following steps:

- Choose a sample point from each small rectangle.
- Find the value of the function $f(x, y) = xy + y$ at each sample point.
- Find the area of each small rectangle.
- Compute the volume of each rectangular prism in the Riemann sum, and add them together.

(a) We'll begin with the first step: **Choose a sample point from each small rectangle.** For this part of the problem, we'll take our sample points to be at the centers of the small rectangles. For convenience, we'll name these sample points $P_{1,1}, P_{1,2}, \dots, P_{3,3}$ as follows:



Your job is to find the coordinates of each of these points. For example, we know that the lower-left corner of $\mathcal{R}$ is $(1, 1.5)$. Also, the width of $\mathcal{R}$ is $5 - 1 = 4$. This means that the width of each small rectangle is $4/3$. Thus, the lower-right corner of the small rectangle containing $P_{1,1}$ is $(1 + 4/3, 1.5) = (7/3, 1.5)$. The x-coordinate of $P_{1,1}$ must therefore be the average of 1 and $7/3$, which is $5/3$. Try to find the y-coordinate of $P_{1,1}$; you should get $7/4$. Now, fill in the rest of this table of coordinates:

$P_{1,3} = \left(\frac{5}{3}, \frac{11}{4}\right)$	$P_{2,3} = \left(\frac{9}{3}, \frac{11}{4}\right)$	$P_{3,3} = \left(\frac{13}{3}, \frac{11}{4}\right)$
$P_{1,2} = \left(\frac{5}{3}, \frac{9}{4}\right)$	$P_{2,2} = \left(\frac{9}{3}, \frac{9}{4}\right)$	$P_{3,2} = \left(\frac{13}{3}, \frac{9}{4}\right)$
$P_{1,1} = (\frac{5}{3}, \frac{7}{4})$	$P_{2,1} = \left(\frac{9}{3}, \frac{7}{4}\right)$	$P_{3,1} = \left(\frac{13}{3}, \frac{7}{4}\right)$

(b) Now we need to do the second step: **Find the value of the function** $f(x, y) = xy + y$ **at each sample point.** For example, *recommend using same denominator for convenience later

$$f(P_{1,1}) = f\left(\frac{5}{3}, \frac{7}{4}\right) = \frac{5}{3} \cdot \frac{7}{4} + \frac{7}{4} = \frac{14}{3}.$$

$$\frac{35}{12} + \frac{21}{12} = \frac{56}{12}$$

All you need to do is fill out the rest of this table of values:

$f(P_{1,3}) = \frac{88}{12}$	$f(P_{2,3}) = \frac{132}{12}$	$f(P_{3,3}) = \frac{176}{12}$
$f(P_{1,2}) = \frac{72}{12}$	$f(P_{2,2}) = \frac{108}{12}$	$f(P_{3,2}) = \frac{144}{12}$
$f(P_{1,1}) = \frac{14}{3} = \frac{56}{12}$	$f(P_{2,1}) = \frac{84}{12}$	$f(P_{3,1}) = \frac{112}{12}$

$f(\frac{5}{3}, \frac{9}{4}) = \frac{5}{3} \cdot \frac{9}{4} + \frac{9}{4} = \frac{45}{12} + \frac{27}{12} = \frac{72}{12}$

$f(\frac{5}{3}, \frac{11}{4}) = \frac{5}{3} \cdot \frac{11}{4} + \frac{11}{4} = \frac{55}{12} + \frac{33}{12} = \frac{88}{12}$

$f(\frac{9}{3}, \frac{7}{4}) = \frac{9}{3} \cdot \frac{7}{4} + \frac{7}{4} = \frac{63}{12} + \frac{21}{12} = \frac{84}{12}$

$f(\frac{9}{3}, \frac{9}{4}) = \frac{9}{3} \cdot \frac{9}{4} + \frac{9}{4} = \frac{81}{12} + \frac{27}{12} = \frac{108}{12}$

$f(\frac{9}{3}, \frac{11}{4}) = \frac{9}{3} \cdot \frac{11}{4} + \frac{11}{4} = \frac{99}{12} + \frac{33}{12} = \frac{132}{12}$

$f(\frac{13}{3}, \frac{7}{4}) = \frac{13}{3} \cdot \frac{7}{4} + \frac{7}{4} = \frac{91}{12} + \frac{21}{12} = \frac{112}{12}$

$f(\frac{13}{3}, \frac{9}{4}) = \frac{13}{3} \cdot \frac{9}{4} + \frac{9}{4} = \frac{117}{12} + \frac{27}{12} = \frac{144}{12}$

$f(\frac{13}{3}, \frac{11}{4}) = \frac{13}{3} \cdot \frac{11}{4} + \frac{11}{4} = \frac{143}{12} + \frac{33}{12} = \frac{176}{12}$

(c) The next step is to **Find the area of each small rectangle.** This is easy because all of the small rectangles are the same size! Find this common area.

$A = \Delta x \, \Delta y = \frac{4}{3} \cdot \frac{1}{2} = \frac{4}{6} = \frac{2}{3}$

(d) The final step is to **Compute the volume of each rectangular prism in the Riemann sum, and add them together.** Remember: the rectangular prisms have the small rectangles as their bases, and the values of f at the sample points as their heights. For example, the rectangular prism corresponding to $P_{1,1}$ has volume

$$(\text{area of small rectangle}) \cdot f(P_{1,1}) = (\text{answer to part c})\frac{14}{3}.$$

Fill out this table with your answers:

$\frac{88}{12} \cdot \frac{2}{3} = \frac{88}{18}$	$\frac{132}{12} \cdot \frac{2}{3} = \frac{132}{18}$	$\frac{176}{12} \cdot \frac{2}{3} = \frac{176}{18}$
$\frac{72}{12} \cdot \frac{2}{3} = \frac{72}{18}$	$\frac{108}{12} \cdot \frac{2}{3} = \frac{108}{18}$	$\frac{144}{12} \cdot \frac{2}{3} = \frac{144}{18}$
$\frac{56}{12} \cdot \frac{2}{3} = \frac{56}{18}$	$\frac{84}{12} \cdot \frac{2}{3} = \frac{84}{18}$	$\frac{112}{12} \cdot \frac{2}{3} = \frac{112}{18}$

Finally, add these numbers together to find the value of the Riemann sum!

$$S_{3,3} = \frac{972}{18} = 54$$

* Note: Another option is to add all the function values together and then multiply by the area of each rectangle:

$$S_{3,3} = \left[f\left(\tfrac{5}{3}, \tfrac{7}{4}\right) + \dots + f\left(\tfrac{13}{3}, \tfrac{11}{4}\right)\right] \cdot \frac{2}{3}$$

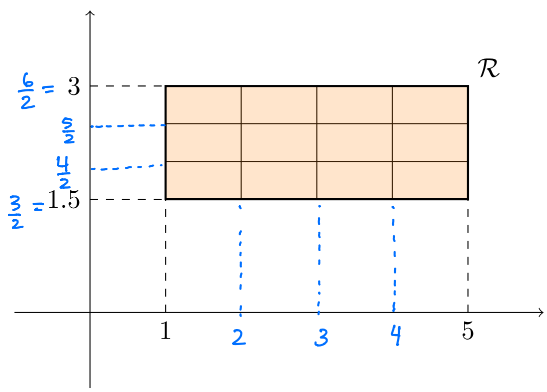
$$= \left[\frac{56}{12} + \frac{72}{12} + \frac{88}{12} + \frac{84}{12} + \frac{108}{12} + \frac{132}{12} + \frac{112}{12} + \frac{144}{12} + \frac{176}{12}\right] \cdot \frac{2}{3}$$

$$= \frac{954}{12} \cdot \frac{2}{3} = 54$$

Problem 2 In this problem, we'll estimate the integral

$$\iint_{\mathcal{R}} xy^2 \, \mathrm{d}A,$$

where $\mathcal{R}$ is still the rectangle $[1,5] \times [1.5,3]$. This time, we'll compute $S_{4,3}$ using sample points from the lower-right corner of each small rectangle. Since we're computing $S_{4,3}$, we are splitting $\mathcal{R}$ in fifths along the x-axis and in quarters along the y-axis, like so:

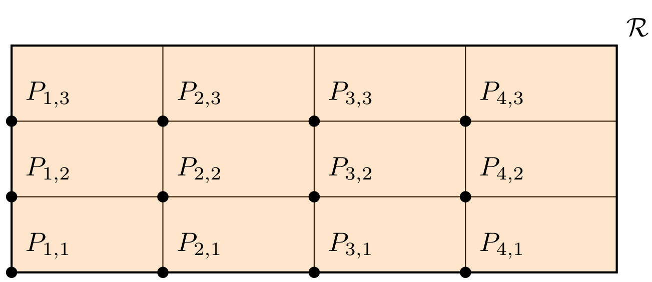


$\Delta x = \frac{5-1}{4} = \frac{4}{4} = 1$

$\Delta y = \frac{3-1.5}{3} = \frac{\frac{3}{2}}{3} = \frac{1}{2}$

We need to do same four steps as before, only this time you won't have any help.

(a) **Choose a sample point from each small rectangle.** This time, we'll pick our sample points to be at the lower-left corner of each small rectangle, like so:



Fill out this table of coordinates:

$P_{1,3} = (1, \frac{5}{2})$	$P_{2,3} = (2, \frac{5}{2})$	$P_{3,3} = (3, \frac{5}{2})$	$P_{4,3} = (4, \frac{5}{2})$
$P_{1,2} = (1, \frac{4}{2})$	$P_{2,2} = (2, \frac{4}{2})$	$P_{3,2} = (3, \frac{4}{2})$	$P_{4,2} = (4, \frac{4}{2})$
$P_{1,1} = (1, \frac{3}{2})$	$P_{2,1} = (2, \frac{3}{2})$	$P_{3,1} = (3, \frac{3}{2})$	$P_{4,1} = (4, \frac{3}{2})$

$f(1, \frac{3}{2}) = 1 \cdot (\frac{3}{2})^2 = 1 \cdot \frac{9}{4} = \frac{9}{4}$

$f(1, \frac{4}{2}) = 1 \cdot (\frac{4}{2})^2 = \frac{16}{4}$

$f(1, \frac{5}{2}) = 1 \cdot (\frac{5}{2})^2 = \frac{25}{4}$

$f(2, \frac{3}{2}) = 2 \cdot \frac{9}{4} = \frac{18}{4}$

$f(2, \frac{4}{2}) = 2 \cdot \frac{16}{4} = \frac{32}{4}$

$f(2, \frac{5}{2}) = 2 \cdot \frac{25}{4} = \frac{50}{4}$

$f(3, \frac{3}{2}) = 3 \cdot \frac{9}{4} = \frac{27}{4}$

$f(3, \frac{4}{2}) = 3 \cdot \frac{16}{4} = \frac{48}{4}$

$f(3, \frac{5}{2}) = 3 \cdot \frac{25}{4} = \frac{75}{4}$

$f(4, \frac{3}{2}) = 4 \cdot \frac{9}{4} = \frac{36}{4}$

$f(4, \frac{4}{2}) = 4 \cdot \frac{16}{4} = \frac{64}{4}$

$f(4, \frac{5}{2}) = 4 \cdot \frac{25}{4} = \frac{100}{4}$

(b) **Find the value of the function $f(x, y) = xy^2$ at each sample point.**

* recommend working w/ same denominator

$f(P_{1,3}) = \frac{25}{4}$	$f(P_{2,3}) = \frac{50}{4}$	$f(P_{3,3}) = \frac{75}{4}$	$f(P_{4,3}) = \frac{100}{4}$
$f(P_{1,2}) = \frac{16}{4}$	$f(P_{2,2}) = \frac{32}{4}$	$f(P_{3,2}) = \frac{48}{4}$	$f(P_{4,2}) = \frac{64}{4}$
$f(P_{1,1}) = \frac{9}{4}$	$f(P_{2,1}) = \frac{18}{4}$	$f(P_{3,1}) = \frac{27}{4}$	$f(P_{4,1}) = \frac{36}{4}$

(c) **Find the area of each small rectangle.**

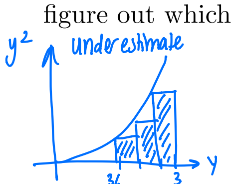
$A = \Delta x \, \Delta y = 1 \cdot \frac{1}{2} = \frac{1}{2}$

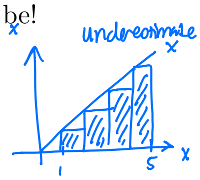
(d) **Compute the volume of each rectangular prism in the Riemann sum, and add them together.** You may use this table to help you if you'd like:

$\frac{25}{4} \cdot \frac{1}{2} = \frac{25}{8}$	$\frac{50}{4} \cdot \frac{1}{2} = \frac{50}{8}$	$\frac{75}{4} \cdot \frac{1}{2} = \frac{75}{8}$	$\frac{100}{4} \cdot \frac{1}{2} = \frac{100}{8}$
$\frac{16}{4} \cdot \frac{1}{2} = \frac{16}{8}$	$\frac{32}{4} \cdot \frac{1}{2} = \frac{32}{8}$	$\frac{48}{4} \cdot \frac{1}{2} = \frac{48}{8}$	$\frac{64}{4} \cdot \frac{1}{2} = \frac{64}{8}$
$\frac{9}{4} \cdot \frac{1}{2} = \frac{9}{8}$	$\frac{18}{4} \cdot \frac{1}{2} = \frac{18}{8}$	$\frac{27}{4} \cdot \frac{1}{2} = \frac{27}{8}$	$\frac{36}{4} \cdot \frac{1}{2} = \frac{36}{8}$

$S_{4,3} =$ $\frac{500}{8} = 62.5$

(e) Let I be the precise value of the integral $\iint_{\mathcal{R}} xy^2 \, dA$. The whole point of Riemann sums is that $S_{4,3}$ should be close to I. Do you think that $S_{4,3}$ will be smaller than, greater than, or exactly equal to I? It is possible to figure out which it'll be!





underestimate

(f) It turns out that $I = 1172934/12412$ (punch this into a calculator). Was your prediction in part e correct? Explain as best you can why you think you were right or wrong.

$I = \frac{189}{2} = 94.5$ our approximation is an underestimate.