Week 10 Worksheet

Problem 1 Use Green's Theorem to find the area of the ellipse $(x/3)^2 + (y/4)^2 = R^2$, where $R > 0$.

By Green's theorem we have

$$\text{Area}(D) = \frac{1}{2}\oint_{\partial D} -y\,dx + x\,dy$$

we parameterise the boundary by

$$\vec{r}(t) = \langle 3R\cos(t), 4R\sin(t) \rangle.$$

$$\text{so } \vec{r}\,'(t) = \langle -3R\sin(t), 4R\cos(t) \rangle$$

$$F(\vec{r}(t)) = \langle -4R\sin(t), 3R\cos(t) \rangle$$

$$F(\vec{r}(t)) \cdot \vec{r}\,'(t) = 12R^2\sin^2(t) + 12R^2\cos^2(t)$$

$$= 12R^2$$

Hence $\text{Area}(D) = \frac{1}{2}\int_0^{2\pi} 12R^2\,dt = 12R^2\pi.$

With alternate notation ("old school")

parameterize the boundary by

$$x = 3R\cos(t), \quad y = 4R\sin(t).$$

Hence $dx = -3R\sin(t)\,dt, \quad dy = 4R\cos(t)\,dt.$

Therefore,

$$\frac{1}{2}\int_{\partial D} -y\,dx + x\,dy = \frac{1}{2}\int_0^{2\pi} +4R\sin(t)\cdot 3R\sin(t)\,dt$$
$$+\, 3R\cos(t)\cdot 4R\cos(t)\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} 12R^2\,dt$$

$$= 12R^2\pi.$$

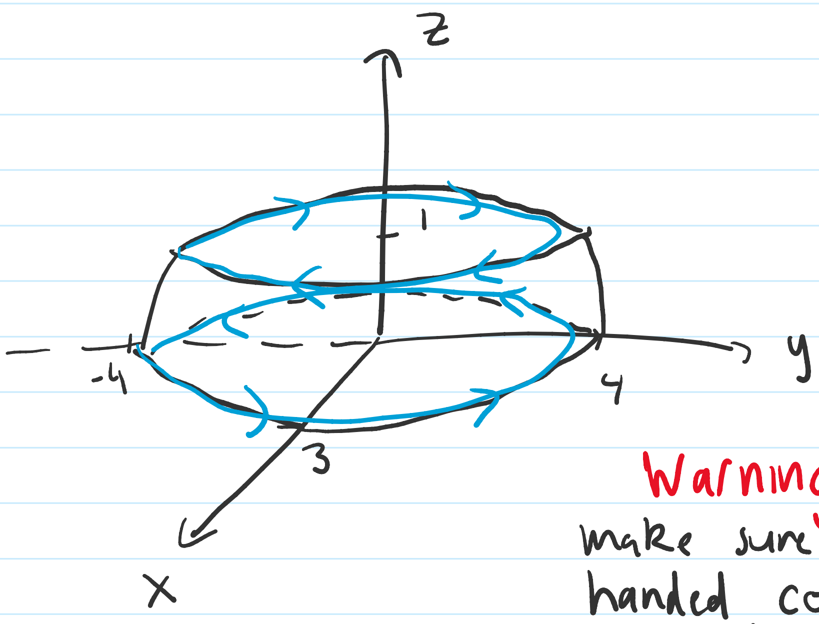
Problem 2 Let $\mathcal{S}$ be portion of the ellipsoid $(x/3)^2 + (y/4)^2 + (z/5)^2 = 1$ lying between the planes $z = 0$ and $z = 1$, oriented with outward-pointing normal vectors.

(a) Draw $\mathcal{S}$ with its boundary orientation.

(b) Let F be the vector field $\langle -y, -z, 1\rangle$. Check that F is the curl of $\langle -y, yz, xz\rangle$.

(c) Use Stokes' Theorem to compute $\iint_{\mathcal{S}} F \cdot d\mathcal{S}$.

a)



Blue boundary

Warning! when drawing make sure you use a right handed coordinate system. Otherwise your drawn boundaries will have the wrong orientation!

(b) $\nabla \times \langle -y, yz, xz\rangle$

$$= \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -y & yz & xz \end{vmatrix}$$

$$= \langle 0 - y, 0 - z, 0 + 1 \rangle = \langle -y, -z, 1 \rangle$$

c) $\iint_S \vec{F} \cdot dS = \int_{\partial S} \langle -y, yz, xz \rangle \cdot d\vec{r}$ by Stokes'.

Let the top circle of the boundary with induced orientation be C_1, the bottom C_2.

parameterise C_1: when $z=1$, $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = \frac{24}{25}$

ie $\left(\frac{5x}{3\sqrt{24}}\right)^2 + \left(\frac{5y}{4\sqrt{24}}\right)^2 = 1.$

Hence we can paramaterise it with (check orientation)

$$\vec{r}(t) = \left\langle \frac{3\sqrt{24}}{5}\cos(t), -\frac{4\sqrt{24}}{5}\sin(t), 1 \right\rangle$$

$$\vec{r}\,'(t) = \left\langle -\frac{3\sqrt{24}}{5}\sin(t), -\frac{4\sqrt{24}}{5}\cos(t), 0 \right\rangle$$

$\vec{G}(\vec{r}(t))$ = something absolutely awful!

At this point you should start looking for a simpler approach. We have that $\vec{F}$ has a vector potential $\vec{F} = \nabla \times \vec{G}$. We have something super nice when this happens:

<u>Theorem</u>: If $\vec{F} = \nabla \times \vec{G}$, then for any two surfaces S, S' such that $\partial S = \partial S'$ (including orientation) then $\iint_S \vec{F} \cdot dS = \iint_{S'} \vec{F} \cdot dS$.

Observe that if I let D_1 be the disk $(x/3)^2 + (y/4)^2 \leq 24/25$, $z=1$ with downward normal and D_2 the disk $(x/3)^2 + (y/4)^2 = 1$, $z=0$.

Then $\partial(D_1 + D_2) = \partial S$.

Hence $\iint_S \vec{F} \cdot dS = \iint_{D_1} \vec{F} \cdot dS + \iint_{D_2} \vec{F} \cdot dS$.

Then $\iint_{D_1} \vec{F} \cdot dS = \iint_{D_1} \langle -y, -z, 1 \rangle \cdot \langle 0, 0, -1 \rangle \, dS$

$$= -\iint_{D_1} dS$$

$$= -\text{Area}(P_1)$$

$$= -12 \cdot \frac{24}{25}\pi \quad \text{by Q1}$$

$$\iint_{D_2} \vec{F} \cdot ds = \iint_{D_2} \langle -y, -z, 1\rangle \cdot \langle 0,0,1\rangle \, ds$$

$$= \text{Area}(D_2)$$

$$= 12\pi.$$

Hence $\iint_S \vec{F} \cdot ds = \frac{12}{25}\pi.$

Problem 3 Let $F = \langle x^3, y^4, z^2\rangle$ and let $\mathcal{S}$ be the portion of the surface $z = x^2 - y^2$ lying inside $x^2 + y^2 = 1$, oriented in the positive z-direction. Use Stokes' Theorem to compute

$$\oint_{\partial S} F \cdot \mathrm{d}r.$$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x^3 & y^4 & z^2 \end{vmatrix}$$

$$= \langle 0, 0, 0\rangle.$$

Hence by Stokes':

$$\int_{\partial S} \vec{F} \cdot dr = \iint_S \nabla \times \vec{F} \cdot dS = 0$$

Problem 4 Let $f(x, y, z)$ be an infinitely differentiable function of 3 variables. Let $\mathcal{S}$ be the level surface $f(x, y, z) = 0$.

(a) Explain why the flux of ∇f through $\mathcal{S}$ must be nonzero, assuming that $\mathcal{S}$ has nonzero surface area.

(b) Use this to show that the vector field $\langle x, y, z\rangle$ does not have a vector potential.

a) we need the additional assumptions that f is not constant and S bounded.

It is a theorem from 32A that if we have a level surface S. i.e, if S is all points $(x,y,z) \in \mathbb{R}^3$ such that $f(x,y,z)=0$.

Then $\nabla f(P)$ is normal to the surface S at P.

For example, if $f(x,y,z) = x^2+y^2+z^2-1$, S is the sphere and $\nabla f = 2\langle x,y,z\rangle$ which is outwards radial and so normal to the sphere.

Since ∇f normal to the surface, suppose the surface is oriented with $\vec{n}$ in the same direction as ∇f. Then $\iint_S \nabla f \cdot dS = \iint_S \nabla f \cdot n \, dS > 0$

since $\nabla f \cdot n > 0$. If $\vec{n}$ otherway, then these are negative. Note: technically ∇f can be zero, but since assuming f is non constant, it can't be zero everywhere.

b) If $\vec{F} = \langle x, y, z \rangle$, take S be the zero level surface of $f(x,y,z) = \frac{1}{2}(x^2+y^2+z^2-1)$. Then $\nabla f = \langle x, y, z \rangle$ and so we see that if we put the outward facing normal on S, from above $\iint_S \vec{F} \cdot ds > 0$.

However, if $\vec{F} = \nabla \times \vec{G}$, then by Stokes' theorem, we have:

$$\iint_S \vec{F} \cdot ds = \iint_S \nabla \times \vec{F} \cdot ds = \int_{\partial S} \vec{G} \cdot d\vec{r} = 0$$

since $\partial S = \emptyset$. This is a contradiction and so $\vec{F}$ can't have a vector potential.

<u>Note:</u> A much easier way to show this is use the fact $\operatorname{div}(\operatorname{curl}(\vec{G})) = 0$.

If $\vec{F} = \nabla \times \vec{G}$, then $\operatorname{div}(\vec{F}) = \operatorname{div}(\nabla \times \vec{G}) = 0$

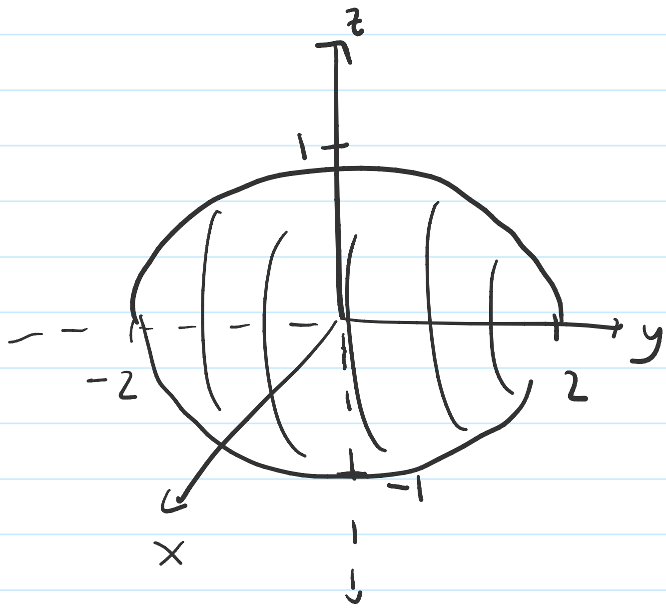
but we have $\operatorname{div}(\vec{F}) = 3$. Contradiction. $\boxtimes$

Problem 5 Let $\mathcal{S}$ be the surface given in spherical coordinates by $\rho = 1 + \sin(\phi)$, oriented outward.

(a) Sketch the surface $\mathcal{S}$.

(b) Use Stokes' Theorem to compute $\iint_{\mathcal{S}} F \cdot dS$, where $F = \langle -4z^3, -2x, -3y^2 \rangle$.

a)



b) Let $\vec{G} = \langle y^3, z^4, x^2 \rangle$.

Then $\nabla \times \vec{G} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y^3 & z^4 & x^2 \end{vmatrix}$

$= \langle -4z^3, -2x, -3y^2 \rangle$

$= \vec{F}$.

Hence by Stokes Theorem

$$\iint_S \vec{F} = \iint_S \nabla \times \vec{G} = \int_{\partial S} \vec{G} \cdot dr = 0 \text{ since } \partial S = \emptyset.$$

Alternatively, Let E be the volume enclosed.

Since $\operatorname{div}(\vec{F}) = 0$, by the divergence theorem we have

$$\iint_S \vec{F} = \iiint_E \operatorname{div}(F)\, dV = 0.$$

∎