MATH 32B Midterm II Feb. 24, 2020

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STUDENT ID NUMBER:

DISCUSSION SECTION NUMBER:

Directions

Answer each question in the space provided. Please write clearly and legibly. Show all of your work—your work must both justify and clearly identify your final answer. No books, notes or calculators are allowed. You must simplify results of function evaluations when it is possible to do so.

For instructor use only

Page	Points	Score
2	12	
3	6	
4	8	
5	10	
6	8	
7	6	
Total:	50	

Let me know if you find any mistakes

1. [6 pts] Compute Div$\vec{\mathbf{F}}$ for the vector field

$$\vec{\mathbf{F}}(x,y,z) = \langle x^2y, \cos z, z + e^{xy} \rangle .$$

If $\vec{\mathbf{F}}$ is describing some sort of motion/flow of particles, is the overall flow near the point $P = (1,-2,3)$ outward (P is a source/point of expansion) or inward (P is a sink/point of contraction)?

$$\text{div} F = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}\cos z + \frac{\partial}{\partial z} z+e^{xy}$$

$$= 2xy + 0 + 1 = 2xy+1$$

at $P=(1,-2,3)$, $\text{Div}\vec{F}(1,-2,3) = 2(1)(-2)+1 = -3$

Since this is negative, the overall flow is inward (a sink).

2. [6 pts] If C is the line segment connecting $(0,1)$ to $(1,0)$ in the xy-plane, compute the line integral $\int_C (y^2 + x^3)\,ds$

parameterise $\vec{r}(t) = (1-t)(0,1) + t(1,0)$

$= (0,1) + t(1,-1) = (t, 1-t)$

$\vec{r}'(t) = (1,-1)$. and $\|r'(t)\| = \sqrt{2}$

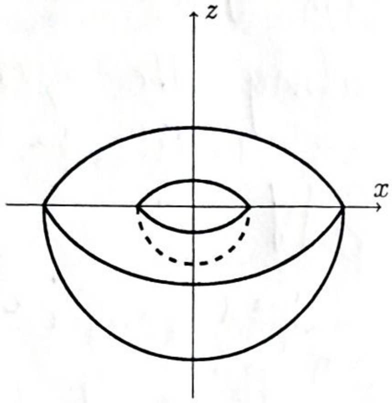
so $\int_C (y^2+x^3)ds = \int_0^1 ((1-t)^2 + t^3)\sqrt{2}\,dt$

$= \sqrt{2}\int_0^1 1 - 2t + t^2 + t^3 dt$

$= \sqrt{2}\left(t - t^2 + \frac{t^3}{3} + \frac{t^4}{4}\right)\Big|_0^1$

$= \sqrt{2}\left(\frac{1}{3} + \frac{1}{4}\right) = \frac{7\sqrt{2}}{12}$

3. Consider a spherical cantaloupe of radius 3, centered at the origin, which is sliced in half. The 'top' half is discarded so that only the 'bottom' half (below the xy-plane) remains. Next, the inner seeds are scooped out leaving a half-spherical 'hole' of radius 1. See the figure below, where the y-axis points into the page (and has been omitted for clarity).



(a) [6 pts] Assuming the cantaloupe has constant mass-density function $\delta(x, y, z) = 6$, find the total mass of the remaining cantaloupe.

In spherical coordinates, the region is given by $1 \le \rho \le 3$, $\frac{\pi}{2} \le \phi \le \pi$, $0 \le \theta \le 2\pi$.

so

$$M = \iiint_E \delta \, dV = \int_0^{2\pi} \int_{\pi/2}^{\pi} \int_1^3 6 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 12\pi \int_{\pi/2}^{\pi} \sin\phi \, d\phi \int_1^3 \rho^2 \, d\rho$$

$$= 12\pi \left[-\cos\phi \right]_{\pi/2}^{\pi} \left[\frac{\rho^3}{3} \right]_1^3$$

$$= 12\pi \left(9 - \frac{1}{3} \right)$$

$$= 4 \cdot 26\pi$$

$$= 104\pi$$

(b) [8 pts] Again assuming that our cantaloupe had constant density $\delta(x, y, z) = 6$, find the (x, y, z)-coordinates of its center of mass. *(HINT: You can use symmetry arguments to determine two of the three coordinates.)*

By symmetry, $x_{cm} = y_{cm} = 0$ since the region is reflective along the x, y axis.

$$z_{cm} = \frac{1}{M} \iiint_E z\delta \, dV$$

$$= \frac{6}{104\pi} \int_0^{2\pi} \int_{\pi/2}^{\pi} \int_1^3 \rho\cos\phi \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{6}{52} \int_{\pi/2}^{\pi} \cos\phi \sin\phi \, d\phi \int_1^3 \rho^3 d\rho$$

$$= \frac{3}{26} \left[\frac{\sin^2\phi}{2} \right]_{\pi/2}^{\pi} \left[\frac{\rho^4}{4} \right]_1^3$$

$$= \frac{3}{26} \left(0 - \frac{1}{2}\right)\left(\frac{3^4}{4} - \frac{1}{4}\right)$$

$$= -\frac{3}{26} \cdot \frac{1}{2} \cdot \frac{80}{4}$$

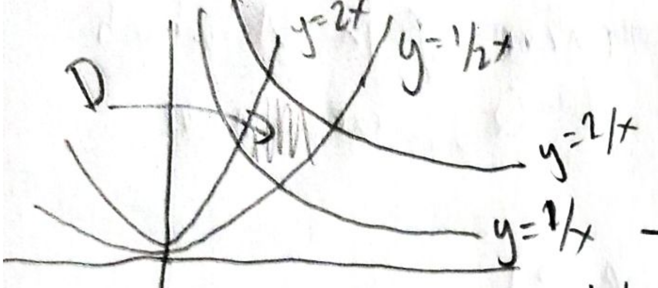
$$= -\frac{3}{26} \cdot 10 = \frac{-15}{13}$$

So center of mass is $\left(0, 0, -\frac{15}{13}\right)$

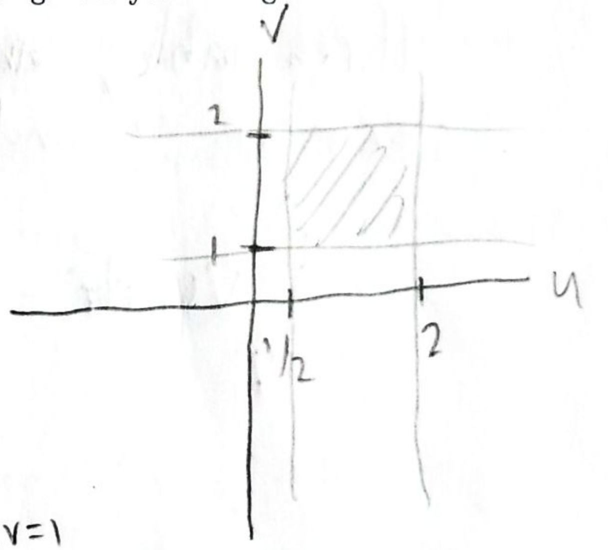
4. [10 pts] Let $\mathcal{D}$ be the domain in the first quadrant of the xy-plane bounded by the following four curves:

$$y = 2x, \quad y = \frac{1}{2}x, \quad y = \frac{2}{x}, \quad y = \frac{1}{x}.$$

Use a change of variables to compute $\iint_{\mathcal{D}} \frac{2\pi^2 y}{x} \sin(\pi xy) \cos\left(\pi \frac{y}{x}\right) dA$. Be sure to draw both the starting domain $\mathcal{D}$ and the 'simplified' domain $\mathcal{D}_0$ resulting from your change of variables.



$\xrightarrow{g}$



let $u = y/x$

$v = xy$

then lines become

$u=2, \ u=1/2, \ v=2, v=1$

$$g(x,y) = (y/x, xy), \quad dg = \begin{pmatrix} -y/x^2 & y \\ 1/x & x \end{pmatrix}$$

$|dg| = -\frac{y}{x} - \frac{y}{x} = -2\frac{y}{x}$. Since in $y,x>0$ region, $\|dg\| = 2\frac{y}{x}$

Hence, $\iint_D 2\pi^2 \frac{y}{x} \sin(\pi xy)\cos(\pi \frac{y}{x}) dA = \pi^2 \int_{1/2}^{2}\int_{1}^{2} \sin(\pi v)\cos(\pi u)\, dv\, du$

$= \pi^2 \left(\int_{1/2}^{2} \cos(\pi u) du\right)\left(\int_1^2 \sin(\pi v) dv\right)$

$= \pi^2 \left[\frac{\sin(\pi u)}{\pi}\Big|_{1/2}^{2}\right]\left[-\frac{\cos(\pi v)}{\pi}\Big|_1^2\right]$

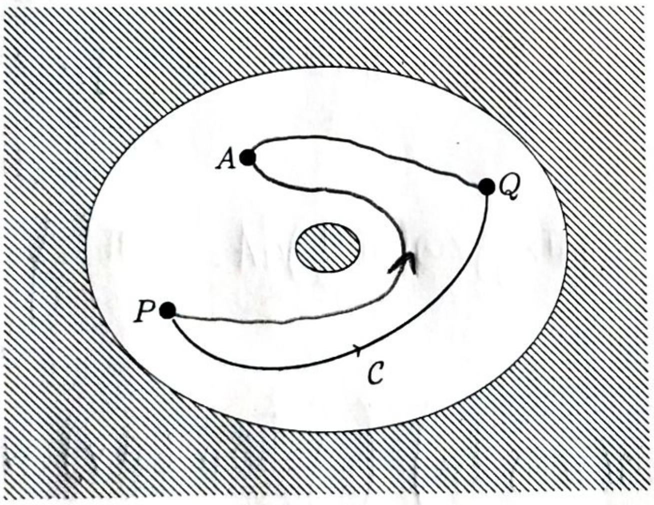
$= \pi^2 \left(-\frac{1}{\pi}\right)\left(-\frac{1}{\pi} + -\frac{1}{\pi}\right) = \pi^2 \cdot \frac{1}{\pi} \cdot \frac{2}{\pi} = 2$

5. [5 pts] The Fundamental Theorem of 1-Variable Integration states that $\int_a^b f'(x)dx = f(b) - f(a)$. State the corresponding Fundamental Theorem of Line Integration. Be very precise!

Given any $\varphi : U \subseteq \mathbb{R}^n \to \mathbb{R}$ which is continuously differentiable, and any curve γ in U which starts at P and ends at Q, we have

$$\int_\gamma \nabla\varphi \cdot d\vec{r} = \varphi(Q) - \varphi(P).$$

6. [3 pts] Suppose that $\mathrm{Curl}(\vec{\mathbf{F}}) = \vec{0}$ throughout the domain pictured below. Draw a curve C' from P to Q that passes through the point A, but also guarantees that $\int_{C'} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.



7. [6 pts] Consider a vector field $\vec{\mathbf{F}}(x, y, z)$ that has the form

$$\vec{\mathbf{F}}(x, y, z) = \langle F_1(x), F_2(y), F_3(z) \rangle,$$

that is, the first component is only a function of x, the second component is only a function of y, and so forth. An example would be something like $\langle x^3, y + e^{\cos y}, \ln z \rangle$.

Assuming that the domain $\mathcal{D}$ of $\vec{\mathbf{F}}$ is simply connected, prove that $\oint_C \vec{\mathbf{F}} \cdot d\vec{r} = 0$ for any closed curve C in $\mathcal{D}$.

Assuming that F_1, F_2, F_3 are differentiable, then $\nabla \times \vec{F} = 0$. Since D simply connected, $\vec{F}$ is conservative and so by the fundamental theorem of line integrals, $\oint \vec{F} \cdot d\vec{r} = 0$

Note, it is possible to show this is true if F_1, F_2, F_3 are just continuous by using the fundamental theorem of calculus to construct a potential function, but the exact proof is a bit too complicated for this course.

(BONUS: Can you explain why the format of $\vec{\mathbf{F}}$ above actually guarantees that $\mathcal{D}$ is simply connected? This is tricky, do not spend time on this if you have not finished all of the other questions on the exam.)

I would disagree. Technically, D is such that every loop is contractible which is half (and the important part) of simply connected.

For instance, $\vec{F} = \langle \frac{1}{x}, 0, 0 \rangle$. The domain D is not connected ($D = \{(x,y,z) \in \mathbb{R}^3 \mid x \neq 0\}$).

However, it is true that every loop fits inside a rectangle that is inside D and so is contractible.