Final Exam

Final

MATH 32B Final Exam Mar. 17, 2020

STUDENT NAME: Ben Szczesny

STUDENT ID NUMBER: ____________

DISCUSSION SECTION NUMBER: ____________

Directions

Answer each question in the space provided. Please write clearly and legibly. Show all of your work—your work must both justify and clearly identify your final answer. No books, notes or calculators are allowed. You must simplify results of function evaluations when it is possible to do so.

For instructor use only

Page	Points	Score
2	8	
3	8	
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10	10	
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Total:	100	

1. [8 pts] Suppose we have a thin linear rod in $\mathbb{R}^3$ with endpoints $(1,1,0)$ and $(0,1,1)$, whose mass density is given by the function

$$\delta_M(x,y,z) = 1 + xyz \quad \text{grams per unit length.}$$

Find the total mass of the rod (in grams).

We integrate δ_M along the line segment $(1,1,0) \to (0,1,1)$.

$r(t) = (1-t, 1, t) \quad 0 \leq t \leq 1$

$r'(t) = (-1, 0, 1)$

$\|r'(t)\| = \sqrt{2}$

$\delta_M(r(t)) = 1 + (1-t)t$

Hence

$$M = \int_\gamma \delta_M \, ds = \int_0^1 (1+(1-t)t)\sqrt{2}\,dt$$

$$= \sqrt{2}\int_0^1 1 + t - t^2 \, dt$$

$$= \sqrt{2}\left[t + \frac{t^2}{2} - \frac{t^3}{3}\right]_0^1$$

$$= \sqrt{2}\left(\frac{3}{2} - \frac{1}{3}\right)$$

$$= \sqrt{2}\left(\frac{7}{6}\right)$$

$$= \frac{7\sqrt{2}}{6}$$

2. [8 pts] Suppose you are waiting for train A and your friend is waiting for train B at the station. Let X denote the wait time for train A, while Y denotes the wait time for train B. Both X and Y are in *minutes*. Suppose that the two wait times have a joint probability density function

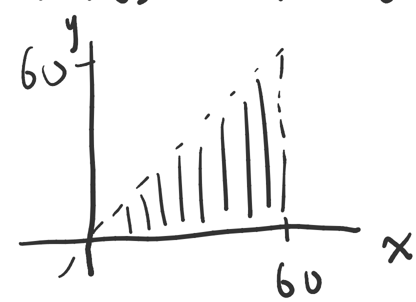
$$p(x, y) = 12e^{-4x-3y}.$$

Suppose you are only willing to wait *one hour* for a train. What is the probability that you will board your train after your friend boards hers? That is to say, what is the probability that train A arrives after train B but before one hour has passed? YOU DO NOT NEED TO SIMPLIFY YOUR ANSWER IN THIS PROBLEM

we want to find

$$\mathbb{P}(Y \le X \le 60) = \iint_D 12e^{-4x-3y}\,dA$$

where $D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le x \le 60\}$.



so $$\iint_D 12e^{-4x-3y}\,dA = \int_{x=0}^{x=60}\int_{y=0}^{y=x} 12e^{-4x-3y}\,dx\,dy$$

$$= 12\int_0^{60} e^{-4x}\int_0^{x} e^{-3y}\,dy\,dx$$

$$= 12\int_0^{60} e^{-4x}\left(-\frac{1}{3}e^{-3y}\Big|_0^x\right)dx$$

$$= 4\int_0^{60} e^{-4x}\left(1 - e^{-3x}\right)dx$$

$$4\int^{60} e^{-4x} \quad -e^{-7x}\,dx$$

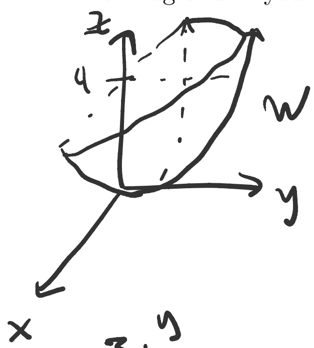
$$= 4\int_0^{60} e^{-4x} - e^{-7x}\,dx$$

$$= 4\left(-\frac{1}{4}e^{-4x} + \frac{1}{7}e^{-7x}\right)\Big|_0^{60}$$

$$= 4\left\{-\frac{1}{4}e^{-240} + \frac{1}{7}e^{-420} + \frac{1}{4} - \frac{1}{7}\right\}$$

$$= \frac{4}{7}e^{-420} - e^{-240} + \frac{3}{7}$$

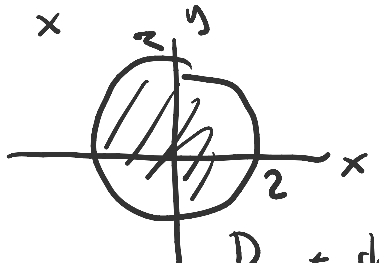
3. [5 pts] Set up the bounds for the integral $\iiint_{\mathcal{W}} y dV$ (but do NOT compute) where $\mathcal{W}$ is the portion of the solid region between the paraboloid $z = x^2 + y^2$ and the plane $z = 4$ where $y \geq 0$. If you use cylindrical or spherical coordinates, be sure to write both the bounds AND the integrand in your chosen coordinate system.



In cylindrical coordinates it is:

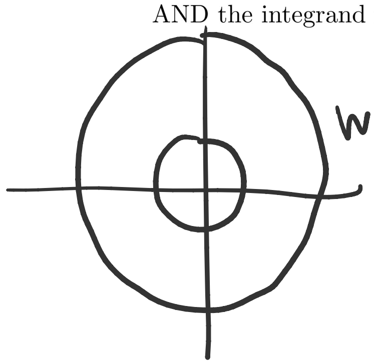
$$\iiint_W y dV = \iint_D \int_0^4 y \, dz \, dA$$

$$= \int_0^{2\pi} \int_0^2 \int_0^4 r^2 \sin\theta \, dz \, dr \, d\theta.$$



D ← shadow in xy plan.

4. [5 pts] Set up the bounds for the integral $\iiint_{\mathcal{W}} z dV$ (but do NOT compute) where $\mathcal{W}$ is the region between the two spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 25$ in the first octant (x, y, z all positive). If you use cylindrical or spherical coordinates, be sure to write both the bounds AND the integrand in your chosen coordinate system.



In spherical coordinates.

$$\iiint_W z dV = \int_0^{2\pi} \int_0^{\pi} \int_2^5 \rho \cos\phi \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_2^5 \rho^3 \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta.$$

5. Let $\vec{\mathbf{F}}(x,y) = \langle y^2 - y, (x^3 - x)\cos(y^2)\rangle$, and let $\mathcal{C}$ be the unit square in the first quadrant with vertices $(0,0), (1,0), (1,1), (0,1)$ traversed counter-clockwise.

(a) [2 pts] Show that $\vec{\mathbf{F}}$ is *not* conservative.

Consider the cross partials. $\vec{F} = \langle F_1, F_2 \rangle$

$$\frac{\partial F_1}{\partial y} = 2y - 1, \quad \frac{\partial F_2}{\partial x} = (3x^2 - 1)\cos(y^2).$$

Hence $\frac{\partial F_1}{\partial y} \neq \frac{\partial F_2}{\partial x}$ and so can't be conservative

(b) [6 pts] Show that, despite the answer above, $\oint_{\mathcal{C}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = 0$. *Hints: Draw $\mathcal{C}$, and think about how $\vec{\mathbf{F}}$ behaves along each side separately.*

when $y=0$, $\vec{F}(x,0) = \langle 0, (x^3 - x)\rangle$.
Hence $\vec{F}$ perpendicular to the line $y=0$ and so the line integral is zero along this.

This also happens when $y=1$.

when $x=0$, $\vec{F}(0,y) = \langle y^2 - y, 0\rangle$ and so $\vec{F}$ perpendicular to the line $x=0$, and the line integral is then zero.
Similarly when $x=1$.

Hence, we see that $\vec{F}$ is always perpendicular to the unit tangent of the curve, so

$$\oint_C \vec{F} \cdot dr = \oint_C \vec{F} \cdot \vec{T}\, ds = \oint_C 0\, ds = 0.$$

6. [10 pts] Evaluate $\iint_{\mathcal{S}} y^2 z dS$ where $\mathcal{S}$ is the surface $x = \sqrt{3}y + \frac{\sqrt{5}}{2}z^2$ with $-1 \le y \le 1$ and $0 \le z \le 1$.

let $y=u$, $z=v$ and $x=\sqrt{3}u+\frac{\sqrt{5}}{2}v^2$.

Hence $\vec{r}(u,v) = \langle \sqrt{3}u + \frac{\sqrt{5}}{2}v^2, u, v \rangle$ $-1 \le u \le 1$, $0 \le v \le 1$ paramaretises the surface.

$$\vec{r}_u(u,v) = \langle \sqrt{3}, 1, 0 \rangle$$

$$\vec{r}_v(u,v) = \langle \sqrt{5}v, 0, 1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 1, -\sqrt{3}, -\sqrt{5}v \rangle$$

$$\|\vec{N}\| = \|\vec{r}_u \times \vec{r}_v\| = \sqrt{1+3+5v^2} = \sqrt{4+5v^2}$$

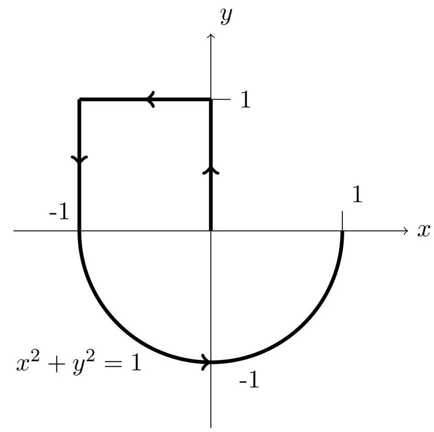
$$\iint_S y^2 z\, dS = \int_0^1 \int_{-1}^1 u^2 v \sqrt{4+5v^2}\, du\, dv$$

$$= \left(\int_{-1}^1 u^2 du\right)\left(\int_0^1 v\sqrt{4+5v^2}\, dv\right)$$

$$= \frac{2}{3}\left[\frac{1}{10}\cdot\frac{2}{3}(4+5v^2)^{3/2}\Big|_0^1\right]$$

$$= \frac{4}{9\cdot 10}\left(9^{3/2} - 4^{3/2}\right) = \frac{2}{9\cdot 5}(27-8) = \frac{38}{45}$$

7. [10 pts] Compute $\int_{\mathcal{C}} \langle e^{\sin y}, xe^{\sin y}\cos y + 6x\rangle \cdot d\overrightarrow{\mathbf{r}}$ where $\mathcal{C}$ is the curve shown below starting at $(0,0)$ and ending at $(1,0)$. *Hints: Green's Theorem*



we complete the curve to a closed curve with line segment γ from $(1,0)$ to $(0,0)$, and let D be the area enclosed.

Now
$$\oint_{C+\gamma} e^{\sin y}dx + (xe^{\sin y}\cos y + 6x)dy$$
$$= \iint_D e^{\sin y}\cos y + 6 - e^{\sin y}\cos y \, dA$$
$$= \iint_D 6\,dA \quad \text{by Green's Theorem.}$$

Hence $\int_C \vec{F}\cdot dr = 6\,\text{Area}(D) + \int_{-\gamma} \vec{F}\cdot dr$

now, we parametrize $-\gamma$ by $x=t,\ y=0 \quad 0\le t\le 1$
and so $\int_{-\gamma} e^{\sin y}dx + (xe^{\sin y}\cos y + 6x)dy \quad \left(\begin{matrix} dx = dt \\ dy = 0\,dt\end{matrix}\right)$

$= \int^1 dt = 1$

$$= \int_0^1 dt = 1.$$

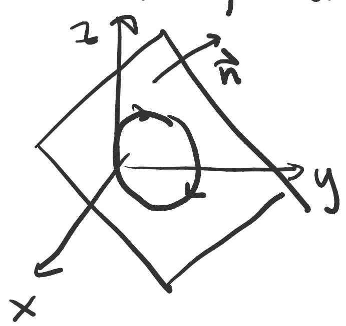
and $\text{Area}(D) = \frac{1}{2}\pi + 1$

Hence $\int_C \vec{F} \cdot dr = 3\pi + 6 + 1 = 3\pi + 7$

8. [6 pts] Compute the line integral $\oint_{\mathcal{C}} \langle y, -2z, 4x\rangle \cdot d\vec{\mathbf{r}}$ where $\mathcal{C}$ is a circle of radius 2 drawn on the plane $x + 2y + 3z = 4$ which is oriented *clockwise* when viewed from above looking downwards.
Hints: Stokes' Theorem

Let $\vec{F} = \langle y, -2z, 4x\rangle$, $\nabla \times \vec{F} = \langle 2, -4, -1\rangle$

We also have $\vec{n} = \frac{1}{\sqrt{14}}\langle 1, 2, 3\rangle$ unit normal to the plane.



Now, $\oint_C \vec{F} \cdot d\vec{r} = -\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS$ by Stokes and reversing orient.

$= \frac{9}{\sqrt{14}} \text{Area}(S) = \frac{36}{\sqrt{14}}\pi$

9. [6 pts] Let $\mathcal{W}$ a 3-dimensional solid cube, with each edge having length 2 (in other words, $\mathcal{W}$ is a solid box with length=width=height=2). As such, the boundary $\partial\mathcal{W}$ consists of six square faces $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_6$, all of which are oriented outwards. Suppose further we have a vector field $\vec{\mathbf{F}}$ that satisfies the following facts:

- $\text{Div}\,\vec{\mathbf{F}} = 3$
- For the first five faces of the cube, the (outward) flux values are all equal:

$$\iint_{\mathcal{S}_1} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_{\mathcal{S}_2} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \cdots = \iint_{\mathcal{S}_5} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = 5.$$

Find the (outward) flux value $\iint_{\mathcal{S}_6} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$ for the sixth and final face.

By divergence theorem we have

$$\iint_{\partial W} \vec{F} \cdot dS = \iiint_W \text{div}\,\vec{F}\, dV = 3\,\text{Vol}(W) = 3 \cdot 8 = 24$$

Now, $\iint_{\partial W} \vec{F} \cdot ds = 25 + \iint_{S_6} \vec{F} \cdot d\vec{s}$. Hence $\iint_{S_6} \vec{F} \cdot ds = -1$.

10. [8 pts] Let $\mathcal{S}$ be the unit sphere in $\mathbb{R}^3$ with outward pointing normal. Show that

$$\iint_{\mathcal{S}} \langle z^y, xy, e^{e^x - e^y} \rangle \cdot d\overrightarrow{\mathbf{S}} = 0.$$

Hints: We can view the unit sphere as the boundary of the solid unit ball in $\mathbb{R}^3$

correction: S centered at origin.

$$\iint_S \langle z^y, xy, e^{e^x - e^y} \rangle \cdot d\vec{S} = \iiint_D 0 + x + 0 \, dV$$

by divergence theorem. where D is solid unit ball.

Then $\iiint_D x \, dV = 0$ by symmetry.

11. [6 pts] It can be proved that, for any scalar function f and vector field $\overrightarrow{\mathbf{F}}$, the curl operator satisfies the following type of product rule:

$$\mathrm{Curl}(f\overrightarrow{\mathbf{F}}) = f\mathrm{Curl}\overrightarrow{\mathbf{F}} + \overrightarrow{\boldsymbol{\nabla}}f \times \overrightarrow{\mathbf{F}}. \quad (\star)$$

Taking this product rule as given (no need to prove it), use Stokes' Theorem to prove that

$$\iint_{\mathcal{S}} (\overrightarrow{\boldsymbol{\nabla}}f \times \overrightarrow{\boldsymbol{\nabla}}g) \cdot d\overrightarrow{\mathbf{S}} = \oint_{\partial\mathcal{S}} (f\overrightarrow{\boldsymbol{\nabla}}g) \cdot d\overrightarrow{\mathbf{r}}$$

for any oriented surface $\mathcal{S}$.

We have $\mathrm{curl}(f\nabla g) = f\,\mathrm{curl}(\nabla g) + \nabla f \times \nabla g$ by $(\star)$

$= \nabla f \times \nabla g$ since $\nabla \times (\nabla g) = 0$

Then by Stokes'

$$\iint_S \mathrm{curl}(f\nabla g) \cdot dS = \oint_{\partial S} (f\nabla g) \cdot dr$$

so $$\iint_S \nabla g \times \nabla f \cdot dS = \oint_{\partial S} f\nabla g \cdot dr.$$

12. [4 pts] Suppose $\mathcal{S}$ is an oriented surface with a parametrization $G(u, v)$, so that the tangent vector fields $\overrightarrow{\mathbf{T}}_u$ and $\overrightarrow{\mathbf{T}}_v$ can be viewed as vector fields on the surface $\mathcal{S}$. Explain why we must always have

$$\iint_{\mathcal{S}} \overrightarrow{\mathbf{T}}_u \cdot d\overrightarrow{\mathbf{S}} = \iint_{\mathcal{S}} \overrightarrow{\mathbf{T}}_v \cdot d\overrightarrow{\mathbf{S}} = 0.$$

Let $\vec{n}$ be the unit normal on S given by orientation. By definition, for any vector field $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS$. Since $\vec{T}_u \cdot \vec{n} = \vec{T}_v \cdot \vec{n} = 0$ we have

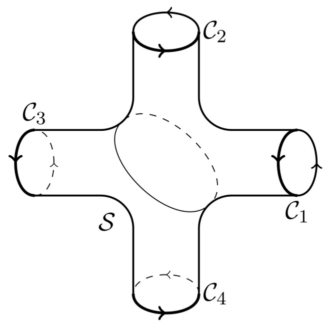
$$\iint_S \vec{T}_u \cdot \vec{n}\, dS = \iint_S \vec{T}_v \cdot \vec{n}\, dS = 0$$

$$\iint_S \vec{T}_u \cdot d\vec{S} = \iint_S \vec{T}_v \cdot d\vec{S} = 0.$$

13. [6 pts] Consider the 'thickened plus sign' surface $\mathcal{S}$ shown below, oriented *outwards*. Suppose we are given the following circulation values for a vector field $\overrightarrow{\mathbf{F}}$ on the various boundary-circles:

$$\oint_{\mathcal{C}_1} \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{r}} = 1, \quad \oint_{\mathcal{C}_2} \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{r}} = \sqrt{2}, \quad \oint_{\mathcal{C}_3} \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{r}} = \pi, \quad \oint_{\mathcal{C}_4} \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{r}} = e.$$

What is the value of $\iint_{\mathcal{S}} \operatorname{Curl} \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{S}}$?



Be careful with orientations! On the top and bottom of the plus, the boundary-circles are oriented counter-clockwise when viewed from above looking downward; on the left and right of the plus, the boundary-circles are oriented counter-clockwise when viewed from the right looking leftward. There is a depth curve drawn on the surface to aid in visualization.

Observe that C_1, C_2 are oriented in the opposite way to the induced orientation.
Hence by Stokes'

$$\iint_S \operatorname{curl}\vec{F} \cdot d\vec{S} = -\oint_{C_1} \vec{F} \cdot dr - \oint_{C_2} \vec{F} \cdot dr + \int_{C_3} \vec{F} \cdot dr + \oint_{C_4} \vec{F} \cdot dr$$

$$= -1 - \sqrt{2} + \pi + e$$

14. For each of these questions, $\overrightarrow{\mathbf{F}}$ denotes a vector field while f denotes a scalar function. Indicate whether the operations described result in vector fields, scalar functions, or do not make sense. Circle your answers (or write down clearly on separate paper). No justification needed.

(a) [2 pts] $\text{Curl}(\text{Curl}(\overrightarrow{\mathbf{F}})) =$

VECTOR FIELD SCALAR FUNCTION

DOES NOT MAKE SENSE

(b) [2 pts] $\text{Div}(\text{Div}(\overrightarrow{\mathbf{F}})) =$

VECTOR FIELD SCALAR FUNCTION

DOES NOT MAKE SENSE (we only defined divergence on $\mathbb{R}^3$)

(c) [2 pts] $\text{Div}(\overrightarrow{\nabla}(f)) =$

VECTOR FIELD SCALAR FUNCTION

DOES NOT MAKE SENSE

(d) [2 pts] $\overrightarrow{\nabla}(\text{Curl}(\overrightarrow{\mathbf{F}})) =$

VECTOR FIELD SCALAR FUNCTION

DOES NOT MAKE SENSE

(e) [2 pts] $\overrightarrow{\nabla}(\text{Div}(\overrightarrow{\mathbf{F}})) =$

VECTOR FIELD SCALAR FUNCTION

DOES NOT MAKE SENSE

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