Problem 1 For the following questions, assume that the vector field $\mathbf{F}$ has a vector potential. That is, $\mathbf{F} = \text{curl}(\mathbf{G})$ wherever it is defined.

(a) Show that for a closed surface (that is a surface S such that $\partial S = \emptyset$) that we have $\iint_S \mathbf{F} \cdot dS = 0$.

(b) Suppose we have two surfaces S, S' such that the boundaries are the same with the same induced orientation. Show that

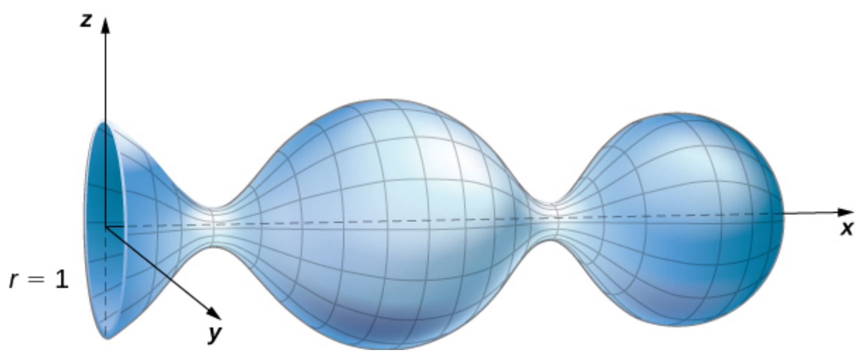
$$\iint_S \mathbf{F} \cdot dS = \iint_{S'} \mathbf{F} \cdot dS.$$

(c) What happens if the induced orientation of the two surfaces in the previous question are not the same?

(d) Show that $\text{div}(\mathbf{F}) = 0$.

(e) Each of the above statements have a corresponding statement when $\mathbf{F}$ has a scalar potential instead (i.e, it is a conservative vector field) and in terms of line integrals instead of surface integrals. What are they?

Problem 2 (From OpenStax) Calculate $\iint_S \text{curl}(\mathbf{F}) \cdot dS$ where $\mathbf{F} = \langle z, x, y \rangle$ and S is the below surface with outwards pointing normal. Bonus: Can you do this without parameterising anything?



Problem 3 (From OpenStax) Calculate the surface integral $\iint_S \mathbf{F} \cdot dS$ where S is the cylinder $x^2 + y^2 = 1$, $0 \leq z \leq 2$ including the top and bottom and $\mathbf{F} = \left\langle \frac{x^3}{3} + yz, \frac{y^3}{3} - \sin(xz), z - x - y \right\rangle$.

Problem 4 Let $\mathbf{F}$ be the inverse square field:

$$\mathbf{F} = \frac{\langle x, y, z \rangle}{r^3}$$

where $r = \sqrt{x^2 + y^2 + z^2}$. Note, this is called the inverse square field since $||\mathbf{F}|| = 1/r^2$.

(a) Show that $\text{div}(\mathbf{F}) = 0$

(b) Let S be the unit sphere around the origin with outwards pointing normal. Show that $\iint_S \mathbf{F} \cdot dS = 4\pi$.

(c) Let E be the volume enclosed by S. The divergence theorem would say that we have

$$\iint_S \mathbf{F} \cdot dS = \iiint_E \operatorname{div}(\mathbf{F}) dV = 0.$$

This contradicts our calculation above. What's going on? *Hint: This is a higher dimensional analogue of the vortex field.*

Problem 5 Consider Problem 1 again, except replace the condition that $\mathbf{F} = \operatorname{curl}(\mathbf{G})$ with $\operatorname{div}(\mathbf{F}) = 0$. When are the statements in Problem (1.a) and Problem (1.b) true? When do you think it's true that $\operatorname{div}(\mathbf{F}) = 0$ implies that $\mathbf{F}$ has a vector potential?

Problem 6 Let S be the portion of the paraboloid $z = 1 - x^2 - y^2$ above the xy-plane and let $\mathbf{F} = \langle y^2 z, x^2 z^2, y \rangle$. Find $\displaystyle\iint_S \mathbf{F} \cdot dS$.

Problem 7 Use the divergence theorem to compute the volume of the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$ (where $a, b, c > 0$).

Problem 8 A function $f : \mathbb{R}^3 \to \mathbb{R}$ is called *harmonic* if $\operatorname{div}(\nabla f) = 0$.

(a) Find an example of a harmonic function and an example of a function which is not harmonic.

(b) Find two harmonic functions whose product is not harmonic.

Problem 9 Let S be the surface defined by $xy + yz - xz = 2$. Given a point on S (for example, $(-1, 0, 2)$) describe how you could find a normal vector to S at that point without parametrizing.

Problem 10 Let S be the surface defined by $x^4 + y^4 + z^4 = 1$, oriented outwards. Show that $\iint_S \langle x^3, y^3, z^3 \rangle \cdot \mathrm{d}S$ is positive.

Problem 11 Let S be the unit sphere in $\mathbb{R}^3$.

(a) Let $\mathbf{F}$ be a vector field. Prove that

$$\iint_S \mathbf{F} \cdot \mathrm{d}S \leq \iint_S \|\mathbf{F}\| \mathrm{d}S$$

(b) Find a vector field $\mathbf{F}$ such that

$$\iint_S \mathbf{F} \cdot \mathrm{d}S = \iint_S \|\mathbf{F}\| \mathrm{d}S$$