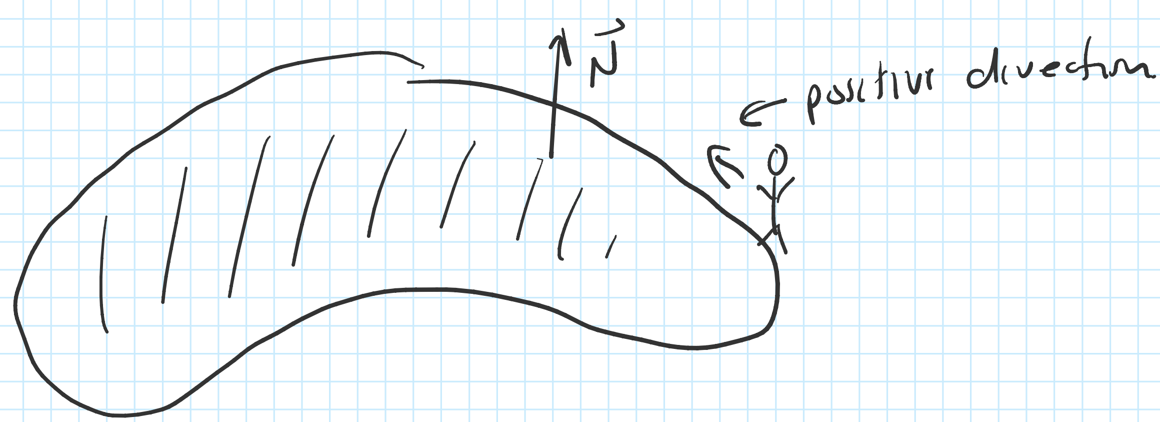
Week 9 Notes

Today: Stoke's Thorem

So Today I want to discuss Stoke's theorem, of which Green's theorem is just a special case. Next week we'll do the divergence theorem.

Boundaries:

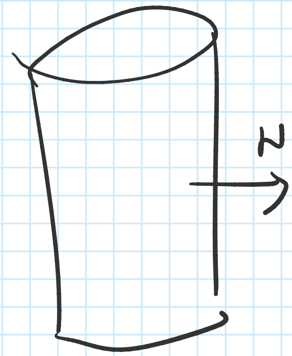
Given a surface S and a given orientation of the normal, we have the boundary ∂S which are curves. Moreover, there is an induced orientation on the boundary when the positive direction is the direction we walk, if we where standing on the boundary in the normal direction and the surface is on our left.



Alternatively, by right hand rule, our thumb is in $\vec{N}$ direction, index finger in the positive direction ann our fist curls into the surface.

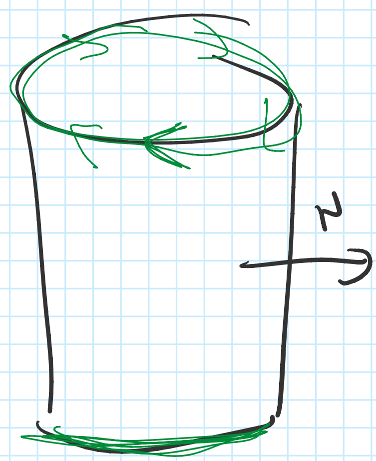
Question: what are the orientations of the boundaries for the following:

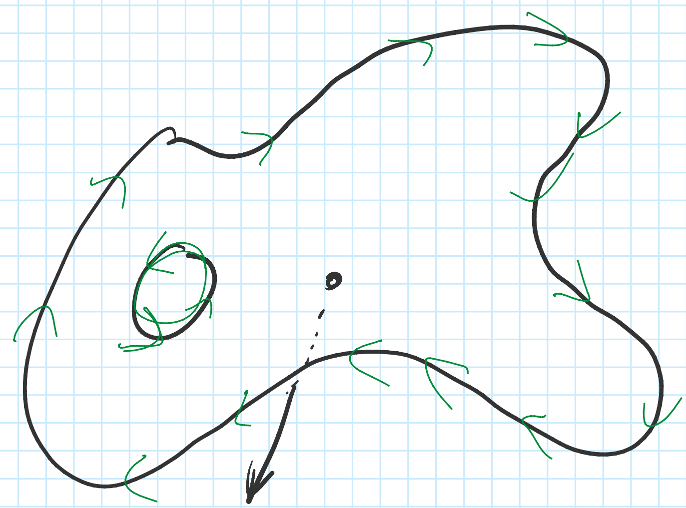
a)



b)

Answer:





Note, if the surface is just a region in $\mathbb{R}^2$ we consider this the xy-plane plane in $\mathbb{R}^3$ and get the orientation by taking the +ve z-axis to be the direction of the normal.

Stoke's Theorem

Given a vector field $\vec{F}$ and surface S in $\mathbb{R}^3$, then we have that

$$\iint_S (\nabla \times \vec{F}) \cdot dS = \int_{\partial S} \vec{F} \cdot dr.$$

(under some niceness assumptions of $\vec{F}$ and S)
Here ∂S has the induced orientation.
So this is a theorem that relates the curl of the vector (a derivative) to it's boundary. But you've seen something like this before!

Massive Tangent: (I think this stuff is cool ok).

So far we have set orientations for curves and surfaces, but what about points?
Suppose we can give a pt either a +ve or -ve orientation.
Also, we have integrals for lines and surfaces, but what about just at pts? We will do so as follows, given pts $P_1, P_2, P_3, \ldots$. Write P_i for positive orientation and $-P_i$ for -ve orientation. We will write "pt integrals" like the following

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$$\int_{P_1 - P_2 + P_3} f\, dp := f(p_1) - f(p_2) + f(p_3).$$

Warning: none of this is standard.

In particular, given a curve C from P to Q, the boundary ∂C is two pts and we will set the induced boundary on them as P -ve and Q +ve.

Then $$\int_{\partial C} f\, dp = \int_{P-Q} f\, dp = f(P) - f(Q).$$

With this notation, the fundamental theorem of line integrals is just the following (remember ∇f is conservative)

$$\int_C \nabla f \cdot ds = \int_{\partial C} f\, dp \quad (= f(P) - f(Q)).$$

remember, ∇ is just a type of derivative and this is in the exact same form as Stokes!

$$\iint_S \nabla \times \vec{F} \cdot dS = \int_{\partial S} \vec{F} \cdot ds.$$

Remember from a few weeks ago we had:

$$
\begin{tikzcd}
\text{Scalar function} \arrow[d, "\text{grad}"', "\nabla"] & \text{pts} \\
\text{vector field} \arrow[d, "\text{curl}"', "\nabla\times"] & \text{lines} \arrow[u, "\partial"'] \\
\text{vector field} \arrow[d, "\text{div}"', "\nabla\cdot"] & \text{surfaces} \arrow[u, "\partial"'] \\
\text{scalar function.} & \text{solids} \arrow[u, "\partial"']
\end{tikzcd}
$$

compare to a similar diagram ↑

- Each row signifies a type of derivative. ie

  first: "pt integrals" over scalar functions

  2nd: line integrals over vector fields

  etc.

Notice there is a pattern here,

You can think of right, then down as $\int_{\partial C} f\,dp$

while down then right as $\int_C \nabla f \cdot ds$, which are equal!

similarly from the second row, right, down gives

$\int_{\partial S} \vec{F} \cdot ds$ while down right gives $\int_S \nabla \times \vec{F} \cdot ds$, which

are equal by stokes.

Question: what do you get by the third row?

Answer: You get

$$\iiint_S \operatorname{div} \vec{F}\, dV = \iint_{\partial S} \vec{F} \cdot dS$$

which is the divergence theorem!

So usually I would have some calculation practice or something equally as boring, but theres nothing new here. Instead I've got some more conceptual questions that we can talk about.

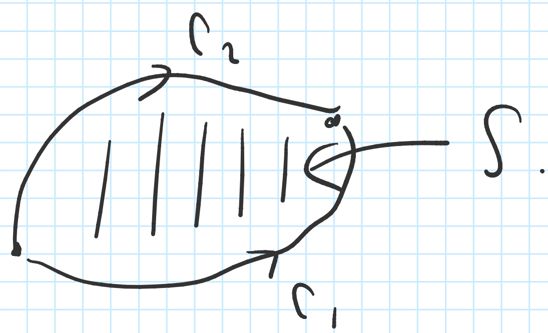
Question:

We know a vector field is conservative if and only if line integrals are path independent. Why do you think it is also true that a vector field with zero curl over a simply connected region (no holes!) is also conservative?

Answer:

If $\vec{F}$ is such that $\nabla \times \vec{F} = 0$, then suppose

If $\vec{F}$ is such that $\nabla \times \vec{F} = 0$, then suppose C_1, C_2 are two paths from a to b. Then $C_1 - C_2$ is a loop. We can fill this in to get a surface S with boundary $C_1 - C_2$.



Now by Stokes theorem $\iint_S \nabla \times \vec{F} \cdot dS = \int_{C_1 - C_2} \vec{F} \cdot ds$

but $\nabla \times \vec{F} = 0 \Rightarrow \int_{C_1 - C_2} \vec{F} \cdot ds = 0$

$$\Rightarrow \int_{C_1} \vec{F} \cdot ds = \int_{C_2} \vec{F} \cdot ds$$

Hence $\vec{F}$ is path independent and so conservative.

Question

If $\vec{F}$ is the curl of some vector field $\vec{G}$ ($\vec{F} = \nabla \times \vec{G}$) ie, $\vec{F}$ has an antiderivative. If S is a surface without a boundary, ie a sphere. What is

$\iint_S \vec{F} \cdot ds = ?$

$$\iint_S \vec{F}\cdot dS = ?$$

Answer:

By Stokes (Assuming $\vec{G}$ nice enough), then

$$\iint_S \vec{F}\cdot dS = \iint_S \nabla\times\vec{G}\cdot dS = \int_{\partial S}\vec{G}\cdot ds = 0.$$

Compare this to the case of conservative integrals! ie $\oint_S \vec{F}\cdot ds = 0$ if $\vec{F} = \nabla f$.

Question: Derive Green's theorem: Given a region D in $\mathbb{R}^2$, and a vector field $\vec{F} = \langle F_1, F_2\rangle$ then

$$\int_{\partial D}\vec{F}\cdot dr = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA$$

Answer: Apply Stokes theorem to $\vec{F}' = \langle F_1, F_2, 0\rangle$.

In particular, we interpret D as being in $\mathbb{R}^3$ on xy-plane and by convention, it has upward facing normal. $n = \langle 0, 0, 1\rangle$.

Since $\nabla\times\vec{F}' = \langle 0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\rangle$, it follows by stokes theorem

$$\int_{\partial D}\vec{F}'\cdot dr = \iint_D \vec{F}'\cdot dS$$

$$\Rightarrow \quad \int_{\partial D} \vec{F} \cdot dr = \iint_D \left\langle 0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle \cdot \langle 0, 0, 1 \rangle \, dA.$$