Method to finding potential functions.

Example: Suppose we have

$\vec{F} = \langle 2xy, x^2+z, y+e^z \rangle$.

if this was conservative, then $\vec{F} = \nabla f$ for some f.

ie $\frac{\partial}{\partial x} f = 2xy \Rightarrow f = \int 2xy\,dx = x^2 y + C(y,x)$

But also, $\frac{\partial}{\partial y} f = F_2$ ie $x^2 + \frac{\partial}{\partial y} C(y,x) = x^2 + z$.

$\Rightarrow \frac{\partial}{\partial y} C(y,x) = z \Rightarrow C(y,x) = \int z\,dy$

$= zy + D(z)$.

Hence $f = x^2 y + zy + D(z)$

but $\frac{\partial}{\partial z} f = F_3$ ie $y + \frac{\partial}{\partial z} D(z) = y + e^z$

so $\frac{\partial}{\partial z} D(z) = e^z \Rightarrow D(z) = \int e^z dz = e^z + \text{constant}$.

Hence $f = x^2 y + zy + e^z$ is a potential for $\vec{F}$.

Scalar Surface Integrals

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$$\int_S f\,dS = \iint_D f(G(u,v))\,\|\vec{N}(u,v)\|\,du\,dv$$

where $\vec{N}(u,v) = \|\frac{\partial G}{\partial u} \times \frac{\partial G}{\partial v}\|$. (or $\|T_u \times T_v\|$)

ie $\vec{N}$ is normal to the surface.

compare this to the line integral case;

$$\int_C f\,ds = \int_a^b f(r(t))\,\|r'(t)\|\,dt$$

<u>Example:</u> we will calculate (in steps)

$\iint_S z(x^2+y^2)\,dS$ where S is the surface parameterised by $G(u,v) = (u\cos v, u\sin v, u)$

$0 \le u \le 1 \quad 0 \le v \le 1$.

<u>Step 1:</u> Find T_u, T_v. ie the partials of G ($T_u = \frac{\partial G}{\partial u}$, $T_v = \frac{\partial G}{\partial v}$, these are tangents to the surface).

<u>Answer:</u>

$$T_u = \frac{\partial G}{\partial u} = (\cos v, \sin v, 1)$$

$$T_v = \frac{\partial G}{\partial v} = (-u\sin v, u\cos v, 0)$$

Step 2: Find $N = T_u \times T_v$.

Answer:

$$N = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} \begin{matrix} i & j \\ \cos v & \sin v \\ -u\sin v & u\cos v \end{matrix}$$

$$= \langle -u\cos v, -u\sin v, u \rangle.$$

Step 3: What is $\int_S z(x^2+y^2)\,dS = ?$

Answer

$$\int_S z(x^2+y^2)\,dS = \iint_D u\left(u^2\cos^2 v + u^2\sin^2 v\right) \|N\|\,du\,dv$$

$$= \int_0^1 \int_0^1 u^3 \sqrt{u^2+u^2}\,du\,dv$$

$$= \sqrt{2} \int_0^1 \int_0^1 u^4\,du\,dv$$

$$= \frac{\sqrt{2}}{5}.$$

Vector surface integral

$$\iint \vec{F} \cdot dS = \iint \vec{F}(r(u,v)) \cdot \vec{N}(u,v)\,du\,dv.$$

$$\iint_S \vec{F} \cdot dS = \iint \vec{F}(G(u,v)) \cdot \vec{N}(u,v)\, du\, dv.$$

compare with vector line integral:

$$\int_C \vec{F} \cdot ds = \int \vec{F}(r(t)) \cdot r'(t)\, dt.$$

Note, we must ensure that $\vec{N}(u,v)$ is pointing in the correct orientation.

Example:

We will calculate $\iint_S \langle -y, z, -x \rangle \cdot dS$ given $G(u,v) = (u+3v, v-2u, 2v+5)$ for $0 \le u \le 1$, $0 \le v \le 1$ and upward pointing normal

Question: what is $N(u,v) = ?$ (with correct orientation.)

Answer:

$$\frac{\partial G}{\partial u} = (1, -2, 0)$$

$$\frac{\partial G}{\partial v} = (3, 1, 2)$$

$$N(u,v) = \begin{vmatrix} i & j & k \\ 1 & -2 & 0 \\ 3 & 1 & 2 \end{vmatrix} \begin{matrix} i & j \\ 1 & -2 \\ 3 & 1 \end{matrix}$$

$$= \langle -4, -2, 7 \rangle$$

This is upward pointing.

<u>Question</u> what is $\iint_S \vec{F} \cdot dS = ?$

<u>Answer:</u>

$$\iint_S \vec{F} \cdot ds$$

$$= \int_0^1 \int_0^1 \langle -v+2u,\ 2v+5,\ -u-3v \rangle \cdot \langle -4, -2, 7 \rangle \, du\,dv$$

$$= \int_0^1 \int_0^1 4v - 8u - 4v - 10 - 7u - 21v \, du\,dv$$

$$= \int_0^1 \int_0^1 -21v - 15u - 10 \, du\,dv$$

$$= -\frac{21}{2} - \frac{15}{2} - 10 = -18 - 10$$

$= -28.$

Question.

Let $\vec{F} = \langle -y, 2x, x+z \rangle$ and S the upper hemisphere of the unit sphere. Parameterize this by spherical coordinates.

$$G(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$$

where $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi/2$.

What is

$$\iint_S (\nabla \times \vec{F}) \cdot dS = ?$$

with outward pointing normal.

Answer:

step 1: Find $\vec{N}$.

$$\frac{\partial G}{\partial \theta} = (-\sin\theta \sin\phi, \cos\theta \sin\phi, 0)$$

$$\frac{\partial G}{\partial \phi} = (\cos\theta \cos\phi, \sin\theta \cos\phi, -\sin\phi)$$

$$\vec{N}(\theta, \phi) = \begin{vmatrix} i & j & k \\ -\sin\theta \sin\phi & \cos\theta \sin\phi & 0 \\ \cos\theta \cos\phi & \sin\theta \cos\phi & -\sin\phi \end{vmatrix}$$

$$= \langle -\cos\theta\sin^2\phi, -\sin\theta\sin^2\phi, -\sin\phi\cos\phi \rangle.$$

so $\vec{N}(\theta,\phi) = \sin\phi\langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle$

(swap sign so outward facing).

<u>step 2</u>; what is $\nabla\times F = ?$

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & 2x & x+z \end{vmatrix}$$

$$= \langle 0, -1, 3\rangle.$$

<u>step 3</u>: calculate integral

$$\iint_S (\nabla\times\vec{F})\cdot dS$$

$$= \int_0^{\pi/2}\int_0^{2\pi} \langle 0,-1,3\rangle\cdot\left(\sin\phi\langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle\right)d\theta\, d\phi$$

$$= \int_0^{\pi/2}\int_0^{2\pi} -\sin\theta\sin^2\phi + 3\sin\phi\cos\phi\, d\theta\, d\phi$$

$$= \int_0^{\pi/2} (\cos 2\pi - \cos 0)\sin^2\phi + 6\pi\sin\phi\cos\phi\, d\phi$$

$$= \int_0^{\pi/2} 6\pi \sin(2\phi)\, d\phi$$

$$= -3\pi \cos(2\phi) \Big|_0^{\pi/2} = 3\pi.$$

<u>Question:</u> (If there is time)

The boundary of the above hemisphere is parameterised by $r(t) = (\cos t, \sin t, 0)$ $0 \le t \le 2\pi$. ($\vec{F} = \langle -y, 2x, x+z \rangle$ as before)

what is $\int_C \vec{F} \cdot ds = ?$

<u>Answer</u>, this is a line integral.

$$r'(t) = (-\sin t, \cos t, 0)$$

$$\vec{F}(r(t)) = \langle -\sin t, 2\cos t, \cos t \rangle.$$

so $$\int_C \vec{F} \cdot dr = \int_0^{2\pi} \vec{F}(r(t)) \cdot r'(t)\, dt$$

$$= \int_0^{2\pi} \langle -\sin t, 2\cos t, \cos t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle\, dt$$

$$= \int_0^{2\pi} \sin^2 t + 2\cos^2 t\, dt$$

$$= \int_0^{2\pi} 1 + \cos^2 t\, dt \qquad \cos 2t = 2\cos^2 t - 1$$

$$= \int_0^{2\pi} 1 + \cos^2 t \, dt \qquad \cos 2t = 2\cos^2 t - 1$$

$$= \int_0^{2\pi} 1 + \frac{1}{2}(\cos 2t + 1) \, dt$$

$$= \frac{3t}{2} + \frac{\sin 2t}{4} \Big|_0^{2\pi}$$

$$= 3\pi$$

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observe that we have

$$\int_C \vec{F} \cdot dr = \int_S (\nabla \times \vec{F}) \cdot dS .$$

where C is the boundary of S. This infact always happens and is called Stoke's Theorem.