Week 7 Notes

line integrals

integrating a scalar function f along a contour C,

$$\int_C f \, ds := \int_a^b f(r(t)) \|r'(t)\| \, dt$$

where $r(t): [a,b] \to \mathbb{R}^n$ is some parameterisation of C.

Example $\int_C \sqrt{1+9xy} \, ds$ over the curve $r(t) = (t, t^3)$ for $0 \le t \le 1$.

we have $r'(t) = (1, 3t^2)$ and so $\|r'(t)\| = \sqrt{9t^4+1}$

Hence
$$\int_C \sqrt{1+9xy} \, ds = \int_0^1 \sqrt{1+9t \cdot t^3} \cdot \sqrt{9t^4+1} \, dt$$
$$= \int_0^1 1+9t^4 \, dt$$
$$= t + \frac{9}{5} t^5 \Big|_0^1$$
$$= 14/5.$$

Question: $\int_C x^2 z \, ds$ where $r(t) = (e^t, \sqrt{2} t, e^{-t})$

for $0 \le t \le 1$.

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Answer: $r'(t) = (e^t, \sqrt{2}, -e^{-t})$

$$\|r'(t)\| = \sqrt{2 + e^{2t} + e^{-2t}}$$

Hence, $\displaystyle\int_C x^2 z \, ds = \int_0^1 e^{2t} \cdot e^{-t} \sqrt{2 + e^{2t} + e^{-2t}} \, dt$

$$= \int_0^1 \frac{e^t \sqrt{2e^{2t} + e^{4t} + 1}}{e^t} \, dt$$

$$= \int_0^1 \sqrt{(e^t + 1)^2} \, dt$$

$$= \int_0^1 e^t + 1 \, dt$$

$$= e^t + t \Big|_0^1 = e + 1 - (1 + 0) = e.$$

Integrating vector fields along contours

Given a vector field $\vec{F} = \langle F_1, F_2, F_3 \rangle$ and curve C, the line integral is defined

as
$$\int_C F \cdot dr := \int_C (F \cdot T)\, ds$$
where T is the unit vector tangent to the curve.

Given parameterisation $r(t)$, we have that
$$\int_C F \cdot dr = \int_a^b F(r(t)) \cdot r'(t)\, dt.$$

Other common notation: (which I usually use)
$$\int_C F_1\, dx + F_2\, dy + F_3\, dz \quad \text{means} \quad \int_C F \cdot dr.$$

Example: Suppose we want to find $\int_C F \cdot dr$ where $F = \langle z, y^2, x \rangle$ and C parameterized by $r(t) = (t+1, e^t, t^2)$ for $0 \le t \le 2$.

Then $\quad r'(t) = (1, e^t, 2t)$, so
$$\int_C F \cdot dr = \int_0^2 \langle t^2, e^{2t}, t+1 \rangle \cdot \langle 1, e^t, 2t \rangle\, dt$$
$$= \int_0^2 t^2 + e^{3t} + 2t^2 + 2t\, dt$$
$$= \int_0^2 e^{3t} + 3t^2 + 2t\, dt$$
$$= \frac{e^{3t}}{3} + t^3 + t^2 \Big|_0^2 = \frac{e^6}{3} - \frac{1}{3} + 12$$

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$$= \frac{1}{3}\left(e^6 + 35\right)$$

In the other notation

$$\int_C F \cdot dr = \int_C z\,dx + y^2\,dy + x\,dz$$

parameterised by $x = t+1$, $y = e^t$, $z = t^2$

so $dx = (t+1)'\,dt \quad dy = (e^t)'\,dt \quad dz = (t^2)'\,dt$

$\Rightarrow \quad dx = dt, \quad dy = e^t\,dt, \quad dz = 2t\,dt$

Hence

$$\int_C z\,dx + y^2\,dy + x\,dz = \int_0^2 t^2\,dt + e^{2t} \cdot e^t\,dt + (t+1)\,2t\,dt$$

$$= \int_0^2 e^{3t} + 3t^2 + 2t\,dt.$$

$$= \frac{1}{3}\left(e^6 + 35\right).$$

Question:

$\int_C (x-y)\,dx + (y-z)\,dy + z\,dz$ on line segment from $(0,0,0)$ to $(1,4,4)$.

<u>Answer</u> The line segment is parameterised by $r(t) = t(1,4,4)$. Hence $r'(t) = (1,4,4)$ and so

$$\int_C \langle x-y, y-z, z\rangle \cdot dr = \int_0^1 \langle t-4t, 0, 4t\rangle \cdot \langle 1,4,4\rangle \, dt$$

$$= \int_0^1 t - 4t + 16t \, dt$$

$$= \int_0^1 13t \, dt$$

$$= \frac{13}{2} t^2 \Big|_0^1$$

$$= \frac{13}{2}.$$

Conservative vector fields

Fundamental theorem of line integrals
(compare with fundamenta theorem of calculus)
Given conservative $\vec{F}$ with antiderivative f. ($\vec{F} = \nabla f$)
a curve C starting at a and ending at b,
then

$$\int \vec{F} \cdot d\ \ldots\ f(b) - f(a)$$

then $\int_C \vec{F} \cdot dr = f(b) - f(a)$.

Example:

Evaluate $\int_C 2xyz\,dx + x^2 z\,dy + x^2 y\,dz$

over the path $r(t) = (t^3, \cos(\frac{\pi t}{2}), e^{2t})$

for $0 \leq t \leq 1$.

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we will show first that this vector field

$\vec{F} = \langle 2xyz, x^2 z, x^2 y \rangle$ is conservative,

by finding a potential function.

$$\int F_1\,dx = \int 2xyz\,dx = x^2yz + C_1(y,z)$$

$$\int F_2\,dy = \int x^2 z\,dy = x^2yz + C_2(x,z)$$

$$\int F_3\,dz = \int x^2 y\,dz = x^2yz + C_3(x,y).$$

Hence $f(x,y,z) = x^2yz$ is such that

$\nabla f = \vec{F}$

$\nabla f = \vec{F}$.

Therefore

$$\int_C \vec{F} \cdot dr = f(r(1)) - f(r(0))$$
$$= f(1, 0, e^2) - f(0, 1, 1)$$
$$= 0$$

Question:

Evaluate $\oint_C \sin x \, dx + z \cos y \, dy + \sin y \, dz$

over the circle $x^2 + y^2 = 1$, $z = 0$.

Answer:

we find a potential function

$$\int \sin x \, dx = -\cos x + C_1(y, x)$$

$$\int z \cos y \, dy = z \sin y + C_2(x, z)$$

$\int \sin y \, dz = z \sin y + C_3(x, y)$

Hence $f(x,y,z) = -\cos x + z \sin y$ is a potential function. $\vec{F} = \nabla f$.

$\therefore \oint_C \vec{F} \cdot dr = 0$. since over a closed curve.

Some last minute things

- I said last week that $\text{curl}(\vec{F}) = 0$ doesn't imply $\vec{F}$ is conservative. I didn't realise the book actually gives the theorem in 17.2 without proof.

So note that if $\vec{F}$ is a vector field over a <u>simply connected domain</u>, then $\nabla \times \vec{F} = 0$ implies that $\vec{F}$ is conservative.

Simply connected means no holes. See the textbook (vortex fields) for an example of what can go wrong if there are holes textbook (Vortex Fields) for an example of what can go wrong if there are holes. Also for the last question in the homework, you should read the conceptual insight at the end of the chapter.