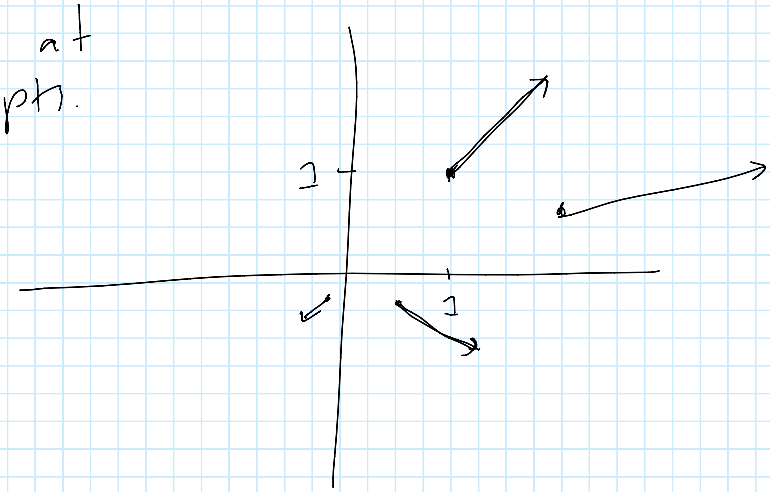
Week 6 Notes

Reminder:

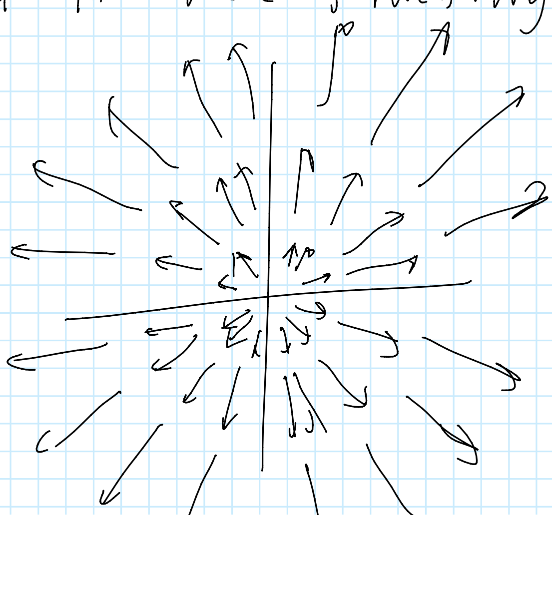
A vector field $\vec{F}$ on $\mathbb{R}^n$ (usually $\mathbb{R}^2$ or $\mathbb{R}^3$ in this course) is a function on $\mathbb{R}^n$ that assigns a point $p \in \mathbb{R}^n$ a vector $v \in \mathbb{R}^n$.

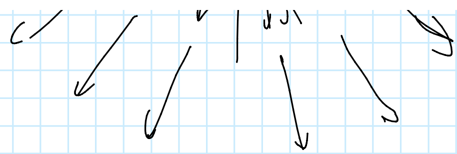
Example: Take the vector field $\vec{F} = \langle x, y \rangle$ on $\mathbb{R}^2$. that is, for each pt $(x,y) \in \mathbb{R}^2$, we attach the vector $\langle x, y \rangle$ onto it.

Picture at a few pts.



and in general it look something like



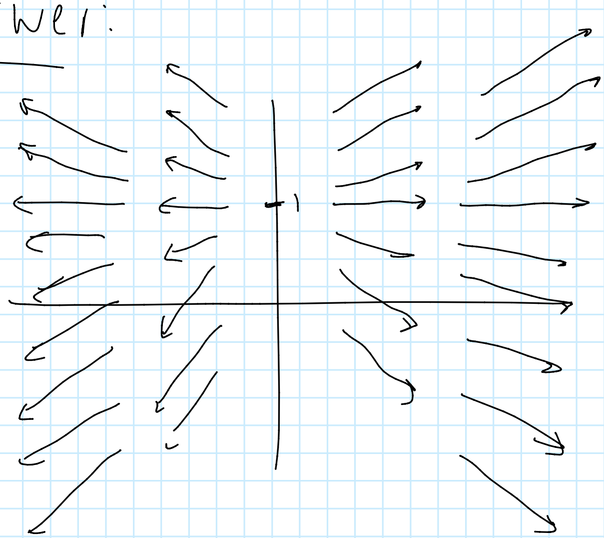


<u>Question:</u> what do they following vector fields look like? If stuck, test a few points.

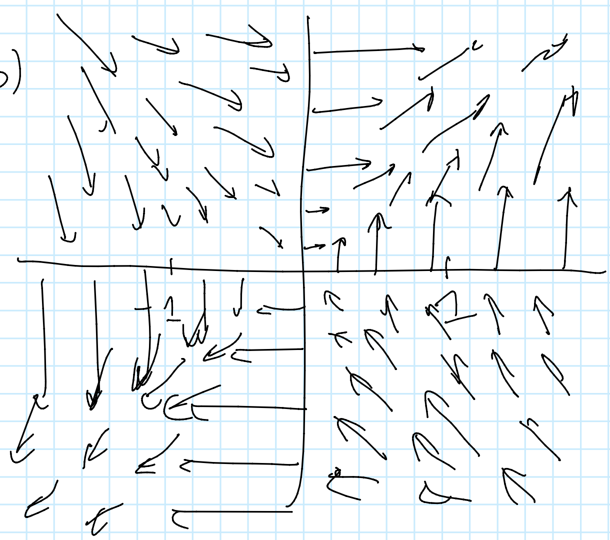
a) $\vec{F} = \langle 2x, y-1 \rangle$ b) $\vec{G} = \langle y, x \rangle$

<u>Answer:</u>

a)



b)



It is sometimes helpful to test pts, or consider what happens on certain lines. ie

In a) on line $y=1$, we get $\vec{F}(x,1) = \langle 2x, 0 \rangle$ so vectors horizontal.

In b) you can try pluggin in lines $x=0$, $y=0$ $x=y$ and $x=-y$.

<u>Reminder</u>

We have two important operations on vector fields in $\mathbb{R}^3$

Given $\vec{F} = \langle F_1, F_2, F_2 \rangle$

Then

$$\operatorname{div}(F) = \nabla \cdot f = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

and curl

$$\operatorname{curl}(F) = \nabla \times f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle.$$

For a vector field $\vec{F} = \langle F_1, F_2 \rangle$ on $\mathbb{R}^2$, we can consider this as a vector field in $\mathbb{R}^3$ by $\vec{F} = \langle F_1, F_2, 0 \rangle$. So we also get div, curl for these vector fields.

Note: div(F) gives us a scalar quantity and curl(F) gives us a vector

Question: Given $\vec{F} = \langle e^y, \sin x, \cos x \rangle$, what is div(F), curl(F)?

Answer:

Answer:

$$\operatorname{div}(F) = \frac{\partial}{\partial x}(e^y) + \frac{\partial}{\partial y}(\sin x) + \frac{\partial}{\partial z}\cos x = 0.$$

$$\operatorname{curl}(F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^y & \sin x & \cos x \end{vmatrix}$$

$$= \left\langle 0, \sin x, \cos x - e^y \right\rangle$$

Reminder:

A vector field $\vec{F}$ is conservative if there exists a function (scalar) f such that $F = \nabla f$. f is called the potential function.

These are essentially antiderivatives! They also share the property that if two exist, they differ by a constant.

Aside: gradient, curl and div form a sort of hierarchy of derivatives

scalar functions on $\mathbb{R}^3$

$\nabla \Big\downarrow$ gradiant

vector fields
$\mathbb{R}^3$

$$\begin{tikzcd} \text{vector fields on } \mathbb{R}^3 \arrow[d, "\nabla\times"', "\text{curl}"] \\ \text{vector fields on } \mathbb{R}^3 \arrow[d, "\nabla\cdot"', "\text{div.}"] \\ \text{scalar functions on } \mathbb{R}^3 \end{tikzcd}$$

where composing two consecutive 'derivatives' gives zero.

ie $\nabla \times (\nabla f) = 0 \quad \nabla \cdot (\nabla \times F) = 0$

Actually, gradiant, div and curl are actually just examples of a more general type of derivative called the exterior derivative.

We have that $\nabla \times (\nabla f) = 0$. In particular, the curl of conservative vector fields is zero.

Question: For the following, which are conservative, which are not? For the conservative vector fields, Find the corresponding potential function.

a) $\vec{F} = \langle ye^{xy}, xe^{xy} \rangle$

b) $\vec{F} = \langle yz^2, xz^2, 2xyz \rangle$.

Warning: In Homework, if a question asks if a vector field is conservative. It is not sufficient to say $\nabla \times F = 0$, hence F conservative. Since we do not yet know this. You can only use curl to show something is <u>not</u> conservative, but not the other way.

Answer:

a) $$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^{xy} & xe^{xy} & 0 \end{vmatrix} = \langle 0, 0, e^{xy} + xye^{xy} - e^{xy} - xye^{xy} \rangle$$

$$= \vec{0}.$$

So now we try and construct a potential function.

We do this by integrating the components of the vector field.

$\int F_1 \, dx = \int ye^{xy} dx = e^{xy} + C_y$ (Here C_y is something that only depends on y)

$\int F_2 \, dy = \int xe^{xy} dy = e^{xy} + C_x$.

$$\int F_2\,dy = \int x e^{xy}\,dy = e^{xy} + C_x.$$

Hence, we see we can join these together and get $f = e^{xy}$ as a potential.

b)

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & xz^2 & 2xyz \end{vmatrix}$$

$$= \langle 2xz - 2xz,\ 2yz - 2yz,\ z^2 - z^2 \rangle$$

$$= \vec{0}.$$

Hence, we try and build a potential.

$$\int F_1\,dx = xyz^2 + C_{yz}$$

$$\int F_2\,dy = yxz^2 + C_{xz}$$

$$\int F_3\,dz = xyz^2 + C_{xy}.$$

Hence $f(x,y,z) = xyz^2$ is a potential.