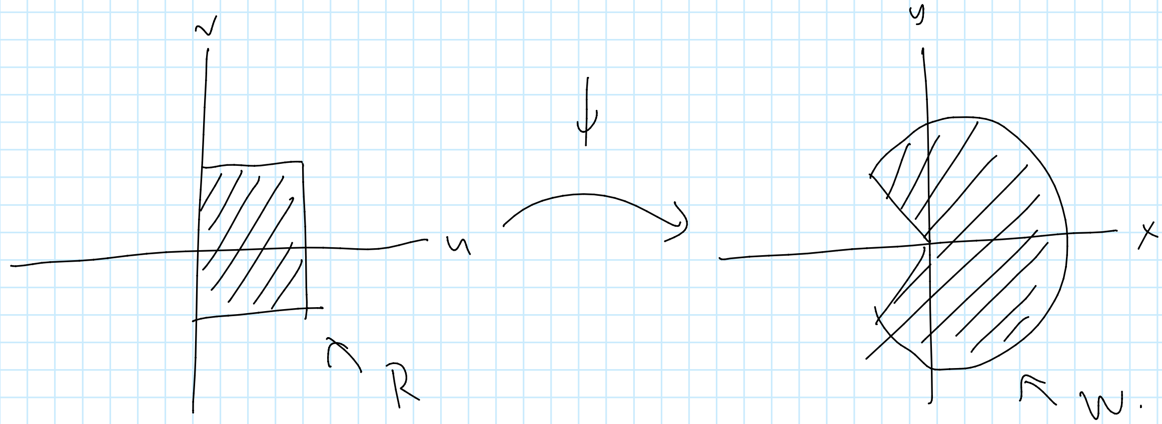
Week 5 Notes

Today: Change of variables formula.

ie, multivariable u-substitution.

Given a region W in $\mathbb{R}^2$, suppose we have a 'nice' function $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ (ie ctsly dif and injective) and another region R in $\mathbb{R}^2$ such that $\phi(R) = W$.

ie, the function ϕ moves R onto W



Then, suppose $\phi(u,v) = (x,y)$, then the change of variables formula is given by:

$$\iint_W f(x,y)\,dx\,dy = \iint_R f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv.$$

where $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} x_u(u,v) & x_v(u,v) \\ y_u(u,v) & y_v(u,v) \end{vmatrix}$ is called the jacobian.

Quick Question: polar coordinates is a function given by $(x,y) = (u\sin v, u\cos v)$. Calculate $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|$

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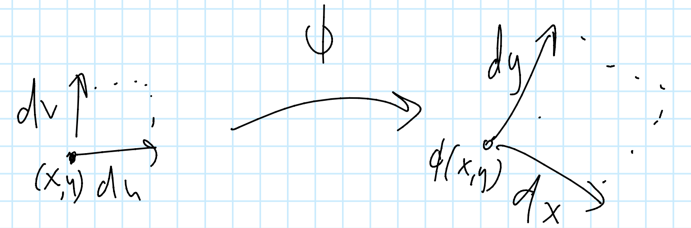
Answer: we have,

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left\|\begin{matrix} x_u & x_v \\ y_u & y_v \end{matrix}\right\| = \left\|\begin{matrix} \sin v & u\cos v \\ \cos v & -u\sin v \end{matrix}\right\| = \left|-u\sin^2 v - u\cos^2 v\right|$$

$= u$. (since in polar, u (which is the r) is positive.

Aside: (If interested).

we can think of $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ as the factor the area of a small rectangle with sides du, dv grows under the function ϕ.



So the jacobian is like a correction factor for the area.

Some warm up Questions:

Question: Let $\phi(u,v) = (u^2, v)$. Then under ϕ, what are the images of the following?

a) The u, v-axis
b) The square $[-1,1]\times[0,1]$.
c) The line segment joining $(0,0)$ to $(1,1)$.

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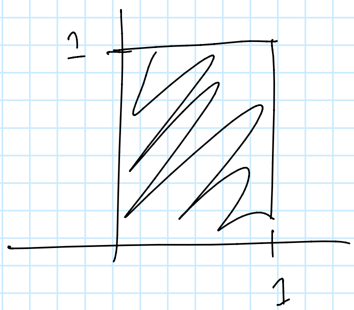
Also, what is the Jacobian?

Answer:

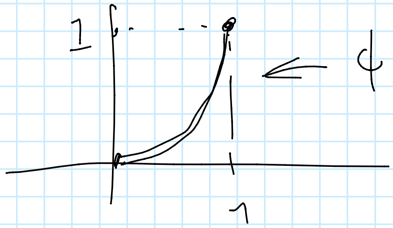
a) when $u=0$, $\phi(0,v)=(0,v)$ and this cuts out the y-axis.
when $v=0$, $\phi(u,0)=(u^2,0)$ which cuts out the positive x-axis.

b) It is not hard to see vertical lines will remain vertical lines. While Horizontal lines also get mapped to horizontal lines, except the x-coordinate is always positive.



$$\phi([-1,1]\times[0,1]) = [0,1]\times[0,1].$$

c) Let us parameterise this line by $(u,v)=t(1,1)=(t,t)$ for $t=0$ to $t=1$. Then $\phi(t,t)=(t,t^2)$. ie, this is a parameterisation of part of a parabola $y=t^2=x^2$ which starts from $(0,0)$ to $(1,1)$. (when $t=0$, to $t=1$)



$\leftarrow \phi(C)$.

Now,

$$J(\phi)=\det\begin{pmatrix}\phi^1_u & \phi^1_v\\ \phi^2_u & \phi^2_v\end{pmatrix} = \det\begin{pmatrix}2u & 0\\ 0 & 1\end{pmatrix} = 2u.$$

$$J(\phi) = \det \begin{pmatrix} \cdots & \cdots \\ \phi^2_u & \phi_v \end{pmatrix} = \det \begin{pmatrix} \cdots & \cdots \\ 0 & 1 \end{pmatrix} = \cdots$$

Question:

Compute the Jacobians of the following:

a) $\phi(u,v) = (v\ln u,\ u^2 v^{-1})$ (for $u, v > 0$)

b) $\phi(u,v) = (ue^v, e^u)$

Answer

a) $$J(\phi) = \det \begin{pmatrix} \phi^1_u & \phi^1_v \\ \phi^2_u & \phi^2_v \end{pmatrix} = \det \begin{pmatrix} \frac{v}{u} & \ln u \\ \frac{2u}{v} & -\frac{u^2}{v^2} \end{pmatrix}$$

$$= -\frac{v}{u}\cdot\frac{u^2}{v^2} - \frac{\ln u \cdot 2u}{v}$$

$$= \frac{-u - 2u\ln u}{v}.$$

b) $$J(\phi) = \det \begin{pmatrix} e^v & ue^v \\ e^u & 0 \end{pmatrix} = -ue^{u+v}.$$

Example: (16.6.29)

Let $D = G(R)$ where $G(u,v) = (u^2, u+v)$ and $R = [1,2]\times[0,6]$.

Calculate $\iint_D y\,dx\,dy$.

Solution:

we first find the Jacobian. we have $G(u,v)=(x,y)$

$x=u^2$, $y=u+v$

$\frac{\partial x}{\partial u}=2u$, $\frac{\partial x}{\partial v}=0$, $\frac{\partial y}{\partial u}=1$, $\frac{\partial y}{\partial v}=1$.

so $J(G)=\det\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix}=\det\begin{pmatrix}2u & 0\\ 1 & 1\end{pmatrix}=2u.$

Note, over R, $u>0$ so $|J(G)|=2u$.

Now,

$$\iint_D y\,dxdy=\iint_R \overbrace{(u+v)}^{y}\,\overbrace{2u\,du\,dv}^{dxdy}$$

$$=\int_0^6\int_1^2 2u^2+2uv\,du\,dv$$

$$=\int_0^6 \frac{2u^3}{3}+u^2v\Big|_1^2\,dv$$

$$=\int_0^6 \frac{16}{3}+4v-\frac{2}{3}-v\,dv$$

$$=\int_0^6 \frac{14}{3}+3v\,dv$$

$$=\frac{14}{3}v+\frac{3}{2}v^2\Big|_0^6=28+54=82$$

Question:

Let $D=G(R)$ where $G(u,v)=(uv^{-1},uv)$ and

$R = [1,2]\times[1,2]$. Calculate $\iint_D (x^2+y^2)\,dx\,dy$.

Answer: we have $x = uv^{-1}$, $y = uv$. so

$x_u = v^{-1}$ $\quad x_v = -uv^{-2}$ $\quad y_u = v$ $\quad y_v = u$.

Hence $J(G) = \det\begin{pmatrix} v^{-1} & -uv^{-2} \\ v & u \end{pmatrix} = uv^{-1} + uv^{-1} = 2uv^{-1}$.

therefore,

$$\iint_D (x^2+y^2)\,dx\,dy = \int_1^2\int_1^2 \left(\frac{u^2}{v^2} + u^2v^2\right) 2\frac{u}{v}\,du\,dv$$

$$= \int_1^2\int_1^2 \frac{2u^3}{v^3} + 2u^3v \,du\,dv$$

$$= \int_1^2 \frac{u^4}{2v^3} + \frac{u^4 v}{2}\Big|_1^2 \,dv$$

$$= \int_1^2 \frac{8}{v^3} - \frac{1}{2v^3} + 8v - \frac{v}{2}\,dv$$

$$= \int_1^2 \frac{15}{2v^3} + \frac{15}{2}v\,dv$$

$$= \frac{-15}{4v^2} + \frac{15}{4}v^2\Big|_1^2$$

$$= \frac{-15}{16} + 15 + \frac{15}{4} - \frac{15}{4}$$

$$= \frac{225}{16}.$$

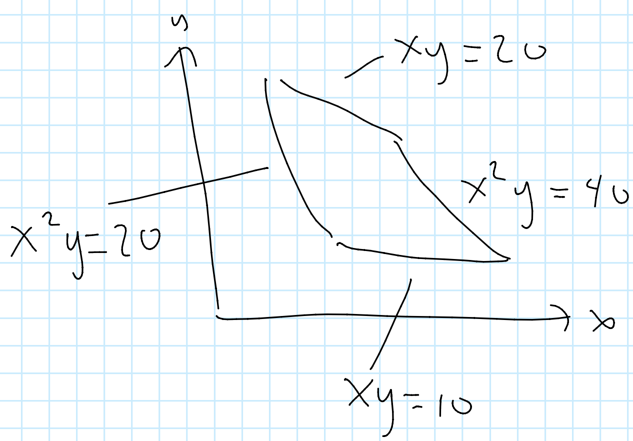
We are not always given D or G beforehand. Usually we a map $F: \mathbb{R}^2 \to \mathbb{R}^2$ from the xy-plane to the uv-plane that either

1) transforms the domain into something nicer
2) turns the function into something we can integrate easier.

or both.

Example: (16.6.39) (Questions)

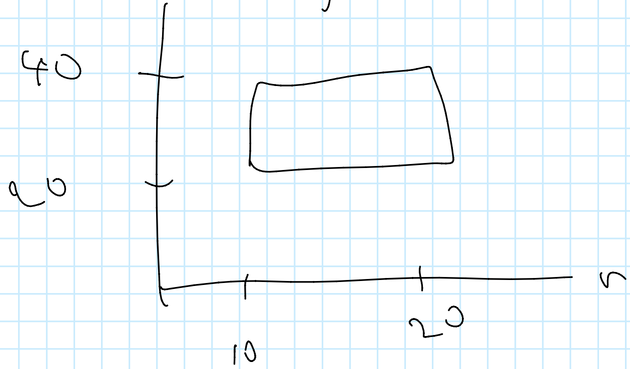
integrate $\iint_D e^{xy}\,dx\,dy$ over the domain



Question: what is the image $F(D)$ where $F(x,y) = (xy, x^2y)$?

Answer: $u = xy$ and $v = x^2y$.

so the bottom line is $u=10$, left is $v=20$
top is $u=20$ and right is $v=40$.



<u>Question</u>: what is the inverse function of F?

<u>Answer</u>, from eq. $\frac{v}{u} = x$, then $u = \left(\frac{v}{u}\right) y$

$$\Rightarrow \quad y = \frac{u^2}{v}.$$

Hence $F^{-1}(u,v) = \left(\frac{v}{u}, \frac{u^2}{v}\right)$

So $G = F^{-1}$ is the wanted function for change of variables.

<u>Question</u>: what is $J(G) = ?$

<u>Answer</u>:

$$J(G) = \begin{vmatrix} -\frac{v}{u^2} & \frac{1}{u} \\ \frac{2u}{v} & -\frac{u^2}{v^2} \end{vmatrix} = \frac{1}{v} - \frac{2}{v} = \frac{-1}{v}.$$

Alternatively, we have $J(G) = J(F^{-1}) = J(F)^{-1}$.

Now, $J(F) = \begin{vmatrix} y & x \\ 2xy & x^2 \end{vmatrix} = yx^2 - 2x^2y = -x^2y.$

and so $J(G) = J(F)^{-1} = \frac{-1}{x^2y} = \frac{-1}{v}$.

Question: Evaluate $\iint_D e^{xy}\,dx\,dy$.

Answer, change of variables with G:

$$\iint_D e^{xy}\,dx\,dy = \int_{10}^{20}\int_{20}^{40} e^u\,|J(G)|\,dv\,du$$

$$= \int_{10}^{20}\int_{20}^{40} e^u v^{-1}\,dv\,du.$$

$$= \left(e^u\Big|_{10}^{20}\right)\left(\ln v\Big|_{20}^{40}\right)$$

$$= (e^{20} - e^{10})(\ln 40 - \ln 20)$$

$$= (e^{20} - e^{10})(\ln 2 + \cancel{\ln 20 - \ln 20})$$