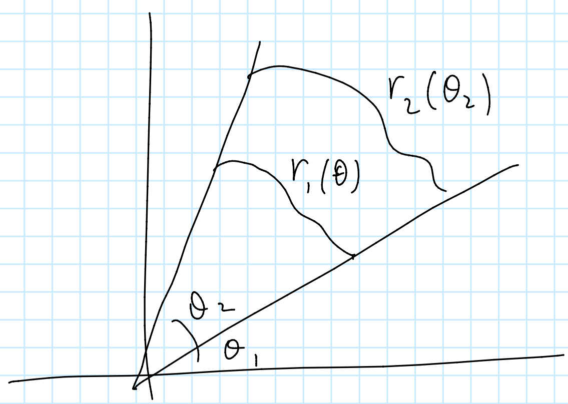
Double integral in polar coordinates.

- when the area lies in some angular sector $\theta_1 \le \theta \le \theta_2$.

want to find the largest and smallest radius for a given angle θ, then region is $\theta_1 \le \theta \le \theta_2$ $r_1(\theta) \le r \le r_2(\theta_2)$



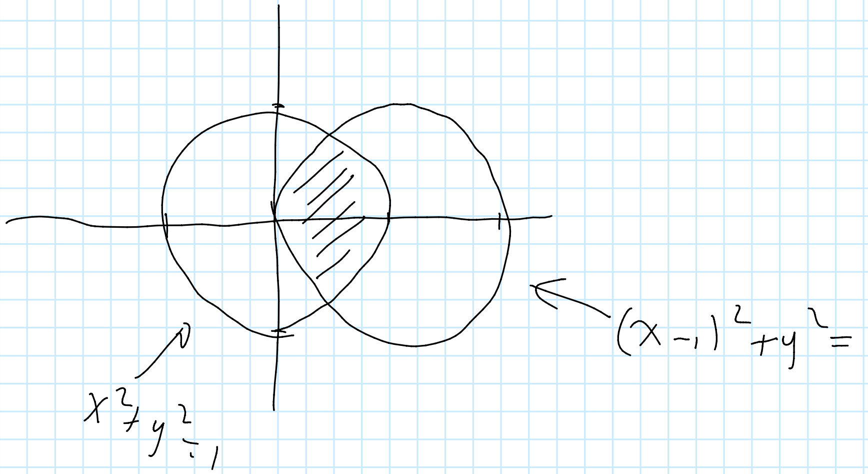
Double integral over region given by:

$$\iint_R f(x,y)\,dA = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

Example:

evaluate $\iint_R y\, dA$ where R is the region bounded by $x^2+y^2 \le 1$ and $(x-1)^2+y^2 \le 1$.

This region is in the intersection of two circles



This shape isn't that easy to write as a radially simple region. It lies within the sector $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. However at different values of θ, the largest value of r is given by the different circles. In particular, the pts of intersection are $\left(\pm\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and, which in polar coordinates is $\left(1, \pm\frac{\pi}{3}\right)$.

so for $-\frac{\pi}{3} \le \theta \le \frac{\pi}{3}$ we have $0 \le r \le 1$

for $-\frac{\pi}{2} \le \theta \le -\frac{\pi}{3}$, $\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$ we have $0 \le r \le 2\cos\theta$

we can join these together into one description by setting
$$g(\theta) = \begin{cases} 1 & \text{if } \theta \in [-\pi/3, \pi/3] \\ 2\cos\theta & \text{if } \theta \in [-\pi/2, \pi/3) \cup (\pi/3, \pi/2] \end{cases}$$

Then the region is given by
$$-\pi/2 \le \theta \le \pi/2, \quad 0 \le r \le g(\theta).$$

Now, $$\iint_R y\,dA = \int_{-\pi/2}^{\pi/2} \int_0^{g(\theta)} r\sin\theta \, r\,dr\,d\theta$$

$$= \int^{\pi/2} \sin\theta \, \frac{r^3}{} \Big|^{g(\theta)} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \sin\theta \, \frac{r^3}{3} \Big|_0^{g(\theta)} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \sin\theta \, \frac{g(\theta)^3}{3} \, d\theta$$

$$= \frac{8}{3} \int_{-\pi/2}^{-\pi/3} \sin\theta \cos^3\theta \, d\theta + \frac{8}{3} \int_{\pi/3}^{\pi/2} \sin\theta \cos^3\theta \, d\theta + \frac{1}{3} \int_{-\pi/3}^{\pi/3} \sin\theta \, d\theta$$

$$= \frac{1}{3} \int_{-\pi/3}^{\pi/3} \sin\theta \, d\theta \quad \text{by symmetry.}$$

$= 0$ since $\sin\theta$ odd.

Note: we could have guessed this from the start as the region is symmetric around the x-axis and y is an odd function.

Question: Using polar coordinates, evaluate

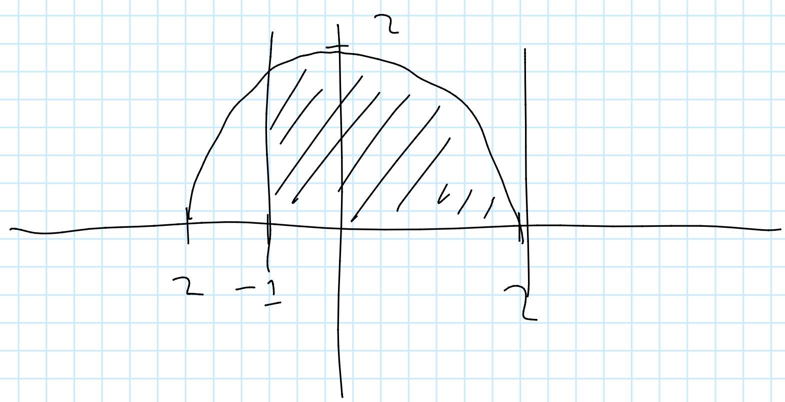
$$\int_{-1}^{2} \int_{0}^{\sqrt{4-x^2}} x^2 + y^2 \, dy \, dx.$$

Answer:

The region we are integrating over is where

$0 \leq y \leq \sqrt{4-x^2}$ $\leftarrow$ upper semicircle

$-1 \le x \le 2 \quad \leftarrow$ strip



this lies in sector $0 \le \theta \le \pi$. Note above, the pt of intersection is $(2, 2\pi/3)$. So it follows the region is given by

for $0 \le \theta \le 2\pi/3$, we have $0 \le r \le 2$

and for $\frac{2\pi}{3} \le \theta \le \pi$, we have $0 \le r \le \sec(\theta - \pi) = -\sec\theta$

(Remember general eq of line: $r = d\sec(\theta - \alpha)$)

$$\text{so } \int_{-1}^{2} \int_{0}^{\sqrt{4-x^2}} x^2 + y^2 \, dy\, dx = \int_0^{2\pi/3} \int_0^2 r^3 \, dr\, d\theta + \int_{\frac{2\pi}{3}}^{\pi} \int_0^{-\sec\theta} r^3 \, dr\, d\theta$$

$$= \int_0^{2\pi/3} 4 \, d\theta + \int_{\frac{2\pi}{3}}^{\pi} \frac{\sec^4(\theta)}{4} \, d\theta$$

Note that $\sec^4 x = \sec^2 x \tan^2 x + \sec^2 x$, so

$$= \frac{8\pi}{3} + \frac{1}{4} \int_{\frac{2\pi}{3}}^{\pi} \sec^2\theta \tan^2\theta + \sec^2\theta \, d\theta$$

$$= \frac{8\pi}{3} + \frac{1}{4} \left[\frac{\tan^3\theta}{3} + \tan\theta \right]_{2\pi/3}^{\pi}$$

$$= \frac{8\pi}{3} + \frac{1}{4}\left(\sqrt{3} + \sqrt{3}\right)$$

$$= \frac{8\pi}{3} + \frac{\sqrt{3}}{2}$$

Correction: In discussion I said the line $x=-1$ was $r=\sec\theta$ since "this cuts out the second line for $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$"

This is wrong as for these values of θ, $\sec\theta < 0$ and this would imply r is negative, which by definition isn't true so actually, $r=\sec\theta$ has no solutions for $\pi/2 < \theta < 3\pi$. Instead, we should have used that $x=-1$ is $x=1$ rotated by π degrees. ie $r = \sec(\theta - \pi) = -\sec\theta$.

Cylinder coordinates:

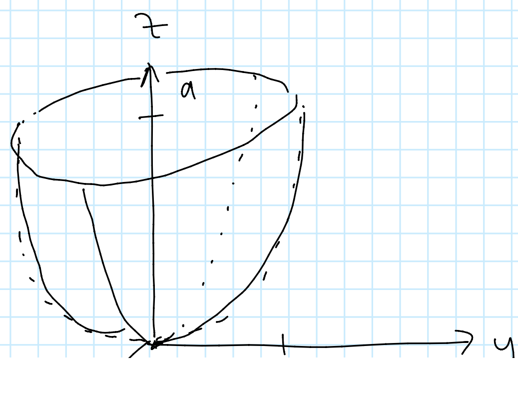
This is just a triple integral where we write the shadow region in polar coordinates.

Question:

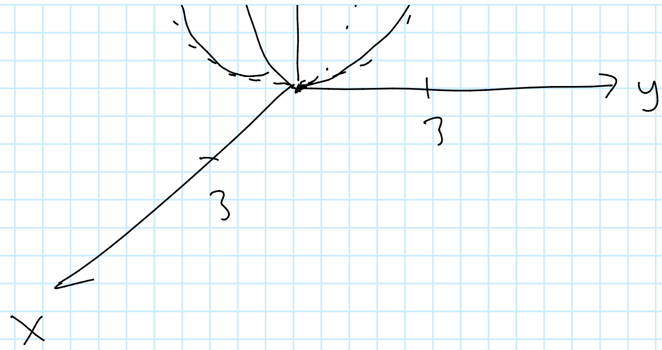
evaluate $\iiint_R z \, dV$ where R is the region given $x^2+y^2 \le z \le 9$.

Answer

The region is



above the parraleboid $x^2+y^2=z$ and below $z=9$.



writing this as a z-simple region, we get

$$\iiint_R z\,dV = \iint_D \int_{x^2+y^2}^{9} z\,dz\,dA$$

where D is the shadow in the xy-plane, which is the circle centred at the origin with radius 3. In polar coordinates, this is given by

$$0 \le r \le 3$$
$$-\pi < \theta \le \pi.$$

Hence

$$\iiint_R z\,dV = \int_0^3 \int_{-\pi}^{\pi} \int_{r^2}^{9} z\,r\,dz\,d\theta\,dr \quad \text{since } x^2+y^2=r^2.$$

$$= \int_0^3 \int_{-\pi}^{\pi} \frac{z^2}{2} r \Big|_{r^2}^{9} d\theta\,dr$$

$$= \int_0^3 \int_{-\pi}^{\pi} \frac{81}{2} r - \frac{r^5}{2}\,d\theta\,dr$$

$$= \pi \int_0^3 81r - r^5\,dr$$

$$= \pi \left[\frac{81r^2}{2} - \frac{r^6}{6} \Big|_0^3 \right]$$

$$= \pi \left[\frac{81 r^2}{2} - \frac{r}{6} \right|_0 \Big)$$

$$= 243\pi.$$

Integration in spherical coordinates.

Remember

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$

(compare to polar coordinates! θ is usual angle in xy-plane)

$$z = \rho \cos\phi$$

Now, we have $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$ after transformation.

Example:

Evaluate $\iiint_W y \, dV$ where W is the region given by

$$x^2 + y^2 + z^2 \le 1, \quad x, y, z \le 0.$$

This region is the part of the sphere in the $(-,-,-)$ octant. Hence, in spherical coordinates it is given by $\frac{\pi}{2} \le \phi \le \pi$, $\pi \le \theta \le \frac{3\pi}{2}$, and $0 \le \rho \le 1$.

Hence,

$$\iiint_W y\,dV = \int_{\pi/2}^{\pi}\int_{\pi}^{3\pi/2}\int_0^1 \rho\sin\theta\sin\phi\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \int_{\pi/2}^{\pi}\sin^2\phi\,d\phi \int_{\pi}^{\frac{3\pi}{2}}\sin\theta\,d\theta \int_0^1 \rho^3\,d\rho.$$

$$= \left(\int_{\pi/2}^{\pi}\frac{1}{2}(1-\cos 2\phi)\,d\phi\right)\left[-\cos\theta\,\Big|_{\pi}^{\frac{3\pi}{2}}\right]\cdot\frac{1}{4}.$$

$$= \left(\frac{1}{2}\phi - \frac{1}{4}\sin 2\phi\,\Big|_{\pi/2}^{\pi}\right)(-1)\cdot\frac{1}{4}$$

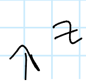
$$= \left(\frac{\pi}{2}-\frac{\pi}{4}\right)\left(-1/4\right)$$

$$= \frac{-\pi}{16}.$$

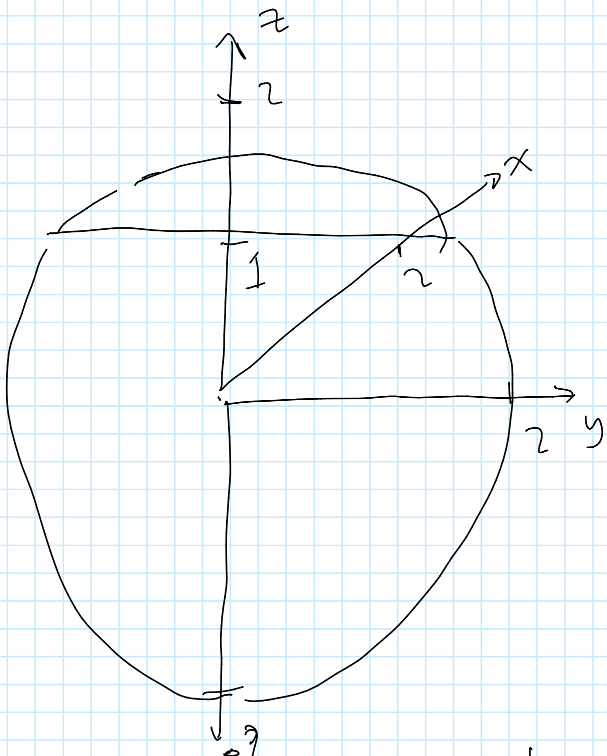
Question:

Evaluate $\iiint_W \sqrt{x^2+y^2+z^2}\,dV$ where W is the region given by $x^2+y^2+z^2 \le 4$, $z \le 1$, $x \ge 0$.

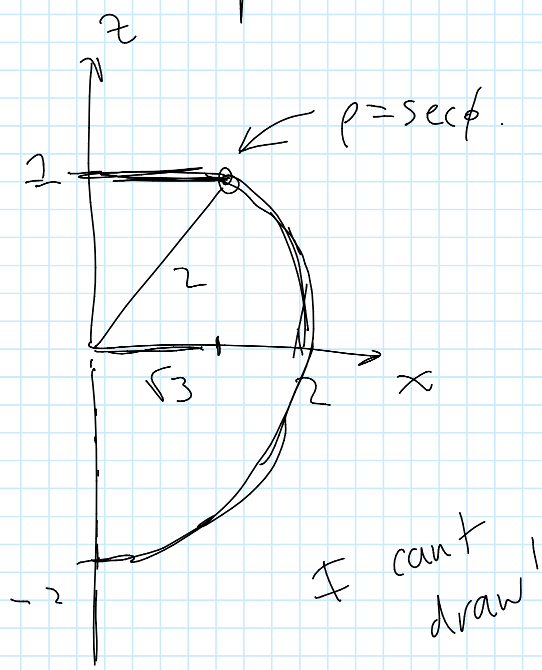
Answer: This is the part of the sphere of radius 2 below $z=1$ and right from $x=0$



shape in zx-plane:



shape in zx-plane:



the intersection of sphere and plane $z=1$ happens when $x^2+y^2+1=4$

$$x^2+y^2=3.$$

Hence, in terms of spherical coordinates, the intersection happens when $\phi=\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)=\pi/3$.

so we get when $\pi/3\le\phi\le\pi$, $0\le\rho\le 2$.

when $0\le\phi\le\pi/3$, the length of a ray is bounded above by $z=1$ ie $\rho\cos\phi=1 \iff \rho=\sec\phi$.

Hence $0\le\rho\le\sec\phi$ in this case.

Hence we get altogether that

$$\iiint_W \sqrt{x^2+y^2+z^2}\,dV=\int_{-\pi/2}^{\pi/2}\left(\int_0^{\pi/3}\int_0^{\sec\phi}+\int_{\pi/3}^{\pi}\int_0^{2}\right)\rho\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\int^{\pi/2}\left(\int^{\pi/3}\frac{\rho^4}{}\Big|^{\sec\phi}+\int^{\pi}\frac{\rho^4}{}\Big|^{2}\right)d\phi\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left(\int_0^{\pi/3} \frac{\rho^4}{4} \Big|_0^{\sec\phi} + \int_{\pi/3}^{\pi} \frac{\rho^4}{4} \Big|_0^{2} \right) d\phi\, d\theta$$

$$= \frac{1}{4} \int_{-\pi/2}^{\pi/2} \left(\int_0^{\pi/3} \sec^4\phi \, d\phi + \int_{\pi/3}^{\pi} 4 \, d\phi \right) d\theta$$

$$= \frac{\pi}{4} \left\{ \int_0^{\pi/3} \sec^2\phi \tan^2\phi + \sec^2\phi \, d\phi + \frac{8\pi}{3} \right\}$$

since $\sec^2 x = \tan^2 x + 1$

$$= \frac{\pi}{4} \left\{ \frac{\tan^3\phi}{3} + \tan\phi \Big|_0^{\pi/3} + \frac{8\pi}{3} \right\}$$

$$= \frac{\pi}{4} \left\{ 2\sqrt{3} + \frac{8\pi}{3} \right\}$$