Week 3

Triple Integrals

Reminders:

- z-simple region consists of points (x,y,z) between two surfaces $z = z_1(x,y)$, $z = z_2(x,y)$ where $z_1(x,y) \le z \le z_2(x,y)$, lying over domain D in xy-plane. So W is defined by

$$(x,y) \in D \qquad z_1(x,y) \le z \le z_2(x,y).$$

In otherwords, all lines parrallel to z-axis that intersect W do so unbroken.

- Triple integrals over W equal to iterated integral

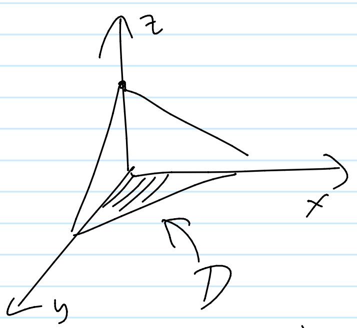
$$\iiint_W f(x,y,z)\, dV = \iint_D \int_{z_1}^{z_2} f(x,y,z)\, dz\, dA.$$

- Nothing special about z above, works for x, y too.

<u>Example</u> evaluate $\iiint_W e^z\, dV$ where W: $x+y+z \le 1$

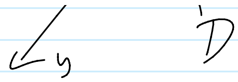
$x \ge 0$, $y \ge 0$, $z \ge 0$.

Now, $x+y+z=1$ is the plane through $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Hence it follows that W is the wedge:



Now, I want z to be the inner integral. So I think for each fixed (x,y), what is the possible range of z-values in this region?

well, it's below the plane (given by



range of z-values in this region?
well, it's below the plane (given by $z \le 1-x-y$) and above xy-plane ($z \geq 0$).
Hence we get $0 \le z \le 1-x-y$.

so $\iiint_W e^z\,dV = \iint_D \int_0^{1-x-y} e^z\,dz\,dA$ where D is the "shadow" of W onto the xy-plane, which we figure out to be given by $0 \le x \le 1$, $0 \le y \le 1-x$.

Hence,

$$\iiint_W e^z\,dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} e^z\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^{1-x} \left(e^{1-x-y} - 1\right) dy\,dx$$

$$= \int_0^1 -e^{1-x-y} - y \Big|_0^{1-x} dx$$

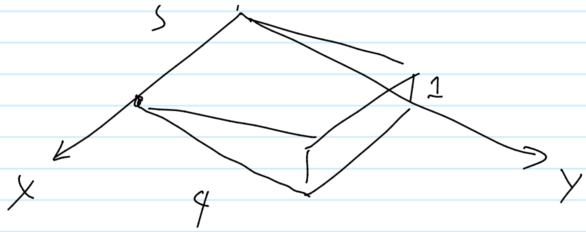
$$= \int_0^1 -1 + e^{1-x} - 1 + x\,dx$$

$$= -e^{1-x} + \frac{x^2}{2} - 2x \Big|_0^1$$

$$= -1 + e + \frac{1}{2} - 2 = e - \frac{5}{2}.$$

Questions

Find $\iiint_W z\,dV$ where W is the shape:



Answer: The top surface is given by $z = \frac{1}{4}y$ (think of corresponding line in yz-plane). Hence it follows for each (x, y), we have the possible z-values $0 \le z \le \frac{1}{4}y$. The shadow of W on the xy-plane is the box $[0,3] \times [0,4]$. Hence

$$\iiint_W z\,dV = \iint_D \int_0^{\frac{1}{4}y} z\,dz\,dA$$

$$= \int_0^3 \int_0^4 \int_0^{1/4 y} z\,dz\,dy\,dx$$

$$= \int_0^3 \int_0^4 \frac{z^2}{2}\Big|_0^{y/4} dy\,dx$$

$$= \int_0^3 \int_0^4 \frac{y^2}{32}\,dy\,dx$$

$$= \int_0^3 \frac{y^3}{96}\Big|_0^4 dx$$

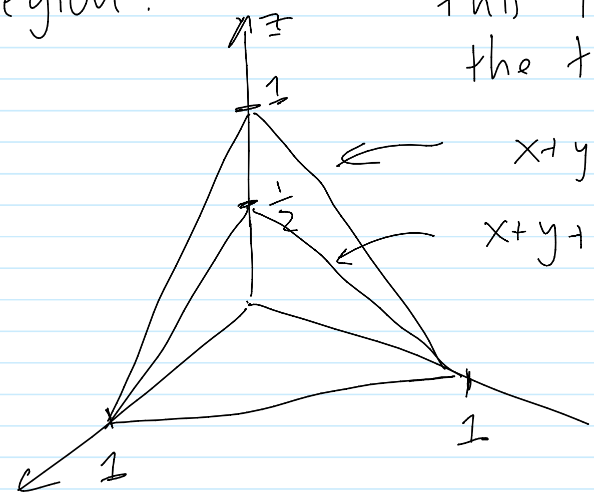
$$= \int_0^3 \frac{2}{3}\,dx = 2.$$

Question

Find volume of the solid in octant $x \ge 0$, $y \ge 0$, $z \ge 0$ bounded by $x + y + z = 1$ and $x + y + 2z = 1$

Answer

Draw the region:



this is the slice between the two planes

$x+y+z=1$.

$x+y+2z=1$

Let W be this solid. For each fixed (x,y), the z-values are below the points s.t $x+y+z=1$ $\Leftrightarrow$ $z=1-x-y$, ie $z \le 1-x-y$.

and they are above points s.t $x+y+2z=1 \Leftrightarrow z=\frac{1}{2}(1-x-y)$ ie $z \geq \frac{1}{2}(1-x-y)$. Hence it follows

$$V=\iiint_W dV = \iint_D \int_{\frac{1}{2}(1-x-y)}^{1-x-y} dz\, dA$$

$$= \int_0^1 \int_0^{1-x} \int_{\frac{1}{2}(1-x-y)}^{1-x-y} dz\, dy\, dx \quad (\text{Note } D \text{ is } \triangle).$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{2}(1-x-y)\, dy\, dx$$

$$= \frac{1}{2}\int_0^1 y - xy - \frac{y^2}{2} \Big|_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 1 - x - x(1-x) - \frac{(1-x)^2}{2}\,dx$$

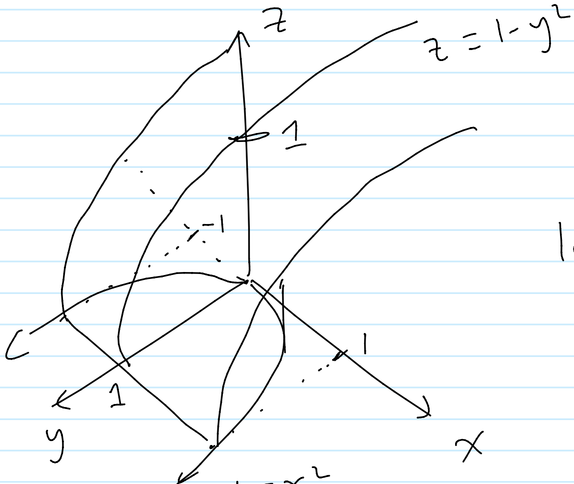
$$= \frac{1}{2}\int_0^1 1-2x+x^2 - \frac{1-2x+x^2}{2}\,dx$$

$$= \frac{1}{2}\int_0^1 \frac{1}{2} - x + \frac{x^2}{2}\,dx = \frac{1}{4}\int_0^1 (1-x)^2\,dx$$

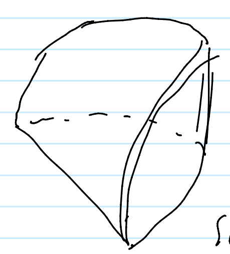
$$= -\frac{1}{12}(1-x)^3\Big|_0^1 = \frac{1}{12}.$$

Question: Let W be the region bounded by $z = 1-y^2$, $y = x^2$ and plane $z=0$. Write volume of W as triple integral in the order $dz\,dy\,dx$; $dx\,dz\,dy$ and $dy\,dz\,dx$.

Answer: We first try and picture this.



looks like:



sort of.

$dz\,dy\,dx$ order:

First look at possible z-values for given (x, y) we see by above $0 \le z \le 1-y^2$

First look at possible z-values for given (x,y) we see by above $0 \le z \le 1-y^2$

Then possible y-values for given x: $x^2 \le y \le 1$

and finally possible x-values: $-1 \le x \le 1$

$$V = \int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y^2} dz\,dy\,dx$$

dxdzdy order:

possible x-values (given (y,z)): $-\sqrt{y} \le x \le \sqrt{y}$

possible z-values: $0 \le z \le 1-y^2$

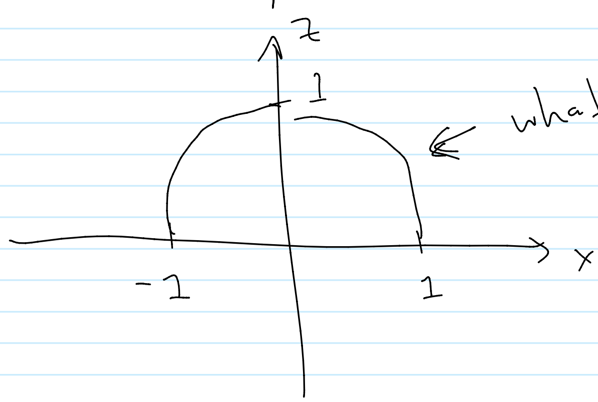
possible y-values: $0 \le y \le 1$

$$\therefore V = \int_{0}^{1}\int_{0}^{1-y^2}\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dz\,dy.$$

dydzdx order:

possible y-values (given (x,z)): $x^2 \le y \le \sqrt{1-z}$

shadow on xz-plane:



what is this curve? It is when $y=x^2$ and $z=1-y^2$ intersect i.e., $z=1-x^4$.

so we get $-1 \le x \le 1$, $0 \le z \le 1-x^4$

Hence $$V = \int_{-1}^{1}\int_{0}^{1-x^4}\int_{x^2}^{\sqrt{1-z}} dy\,dz\,dx.$$

checking these give same volume: (not part of Q).

$$\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y^2} dz\,dy\,dx = \int_{-1}^{1}\int_{x^2}^{1} 1-y^2\,dy\,dx$$

$$= \int_{-1}^{1} y-\frac{y^3}{3}\Big|_{x^2}^{1} dx$$

$$= \int_{-1}^{1} \frac{2}{3} - x^2 + \frac{x^6}{3}\,dx$$

$$= 2\left(\frac{2}{3}x - \frac{x^3}{3} + \frac{x^7}{21}\right)\Big|_0^1$$

$$= 2\left(\frac{1}{3}+\frac{1}{21}\right) = 16/21$$

$$\int_{0}^{1}\int_{0}^{1-y^2}\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dz\,dy = \int_{0}^{1}\int_{0}^{1-y^2} 2\sqrt{y}\,dz\,dy$$

$$= 2\int_0^1 y^{1/2} - y^{5/2}\,dy$$

$$= 2\left(\frac{2}{3}y^{3/2} - \frac{2}{7}y^{7/2}\right)\Big|_0^1$$

$$= 2\left(\frac{2}{3}-\frac{2}{7}\right) = \frac{2(14-6)}{21}$$

$$= 16/21$$

$$\int_{-1}^{1}\int_{0}^{1-x^4}\int_{x^2}^{\sqrt{1-z}} dy\,dz\,dx = \int_{-1}^{1}\int_{0}^{1} \sqrt{1-z} - x^2\,dz\,dx$$

$1-x^4$

$$\int_{-1}\int_{0}\int_{x^2} \cdots = \int_{-1}\int_{0} \cdots$$

$$= \int_{-1}^{1} -\frac{2}{3}(1-z)^{3/2} - x^2 z \Big|_{0}^{1-x^4} dx$$

$$= \int_{-1}^{1} -\frac{2}{3}x^6 - x^2(1-x^4) + \frac{2}{3}\, dx$$

$$= \int_{-1}^{1} -\frac{2}{3}x^6 - x^2 + x^6 + \frac{2}{3}\, dx$$

$$= 2\int_{0}^{1} \frac{1}{3}x^6 - x^2 + \frac{2}{3}\, dx$$

$$= 2\left(\frac{1}{21} - \frac{1}{3} + \frac{2}{3}\right)$$

$$= 2\left(\frac{1}{21} + \frac{1}{3}\right)$$

$$= 16/21$$

So everything checks out.

Polar Coordinates:

- The transformation

$$(x,y) \longmapsto \left(\sqrt{x^2+y^2},\ \tan^{-1}\left(\frac{y}{x}\right)\right) \ (= (r,\theta))$$

is the mapping from rectangular to polar coordinates

the inverse transformation is

$(r, \theta) = (r\cos\theta, r\sin\theta)$

Quick Question: describe what the following eq. look like.

a) $r = 2$ c) $r = 2\sec\theta$

b) $\theta = 2$ d) $r = 2\operatorname{cosec}(\theta)$

Answer:

a) circle radius two c) vertical line, $x = 2$.

b) line from origin d) horizontal line $y = 2$

Question: convert the equation $r = 2\sin\theta$ to an equation in rectangular coordinates

Answer: $r = 2\sin\theta \iff r^2 = 2r\sin\theta$ as $r \neq 0$.

and as $r^2 = x^2 + y^2$

$r\sin\theta = y$

Therefore equation becomes $x^2 + y^2 = 2y$

and completing the square $x^2 + (y-1)^2 = 1$

so circle radius 1 centred at $(0,1)$.