# Week 2 Notes

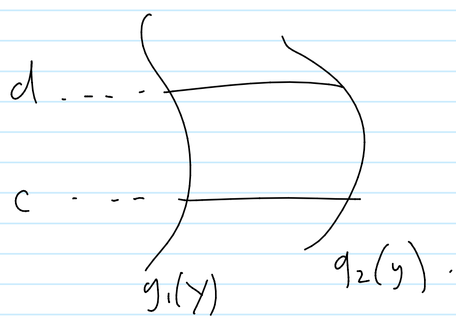
- Tentative O/H: 2-3pm Thursday MS 3957.
- Email: Ben.szczesny@math.ucla.edu.
- H/W due friday 16.1, 16.2 questions.

## More complicated Double integrals:

- A region $D$ is:
  - vertically simple if between two graphs $y = g_1(x)$ $y = g_2(x)$


$$a \leq x \leq b$$
$$g_1(x) \leq y \leq g_2(x)$$

  - horizontally simple if between two graphs $x = g_1(y)$ $x = g_2(y)$.




$$c \leq y \leq d$$
$$g_1(y) \leq x \leq g_2(y).$$

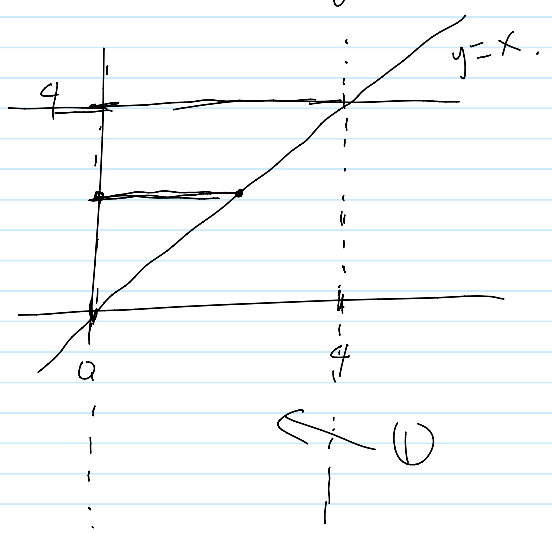
- Most regions are both, and we can convert between the two.

<u>Example:</u> Take the vertically simple region given by

$$0 \leq x \leq 4$$
$$x \leq y \leq 4$$

and write it as a horizontally simple region.

First sketch region:




① $0 \leq x \leq 4$ tells us the region is in this vertical strip.

② $x \leq y \leq 4$ tells us that $y$ is bounded above by grap $y=4$. and below by $y=x$.

③ For a horizontally simple region, we first want to find the range of possible $y$ values.
In this case $0 \leq y \leq 4$.

④, For each given $y$ in this range, we want to now find possible $x$ values. We observe that the largest is given by $x=y$ and smallest by $x=0$. Hence

$$0 \leq x \leq y.$$

Hence region given by $0 \leq y \leq 4$, $0 \leq x \leq y$.

## Questions:

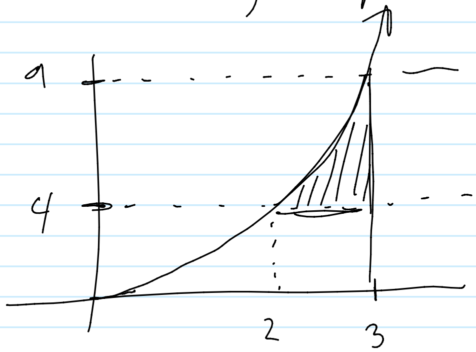
For the following regions, identify if it is written as a horizontally/vertically simple region and rewrite as the opposite.

(a) $4 \leq y \leq 9$, $\sqrt{y} \leq x \leq 3$

(b) $2 \le y \le \sqrt{x}$, $4 \le x \le 9$

Answer:

(a) Horizontally simple. (The bounds of $y$ don't depend on $x$)



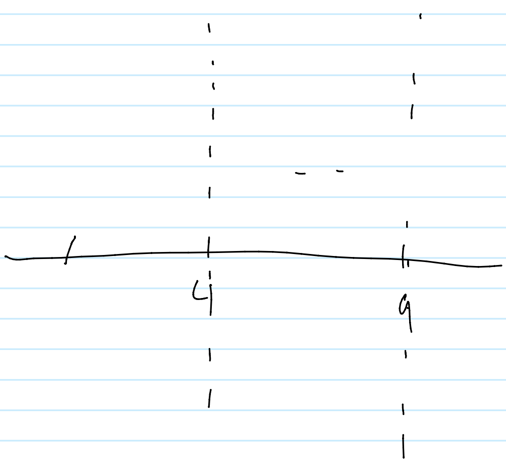
note lower bound of $x$ is $\sqrt{y} = x \iff y = x^2$

upper bound $x = 3$

so range of $x$: $2 \le x \le 3$

and " " $y$: $4 \le y \le x^2$

(b) Vertically simple (bounds of $x$ don't depend on $y$).

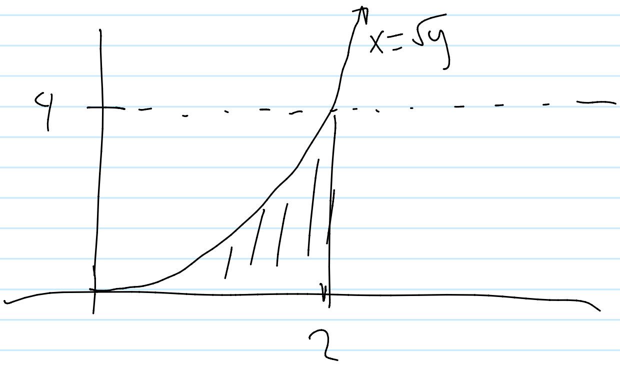


Example: (change of order of integration)

$$\int_0^4 \int_{\sqrt{y}}^2 \sqrt{4x^2 + 5y}\, dx\, dy.$$

we have region given by

$\sqrt{y} \le x \le 2, \quad 0 \le y \le 4.$




so we see $0 \le x \le 2$ and for fixed $x$,

$0 \le y \le x^2$ (as $x = \sqrt{y} \Leftrightarrow y = x^2$).

Hence rearranging gives

$$\int_0^2 \int_0^{x^2} \sqrt{4x^2+5y}\, dy\, dx = \int_0^2 \frac{2}{15}(4x^2+5y)^{3/2} \Big|_0^{x^2} dx$$

$$= \int_0^2 \frac{2}{15}(9x^2)^{3/2} - \frac{2}{15}(4x^2)^{3/2}$$

$$= \int_0^2 \frac{2}{15} \cdot 27x^3 - \frac{2}{15} \cdot 8x^3\, dx$$

$$= \frac{27}{30}x^4 - \frac{4}{15}x^4 \Big|_0^2$$

$$= 16\left(\frac{27}{30} - \frac{4}{15}\right)$$

$$= \frac{16 \cdot 19}{30}.$$

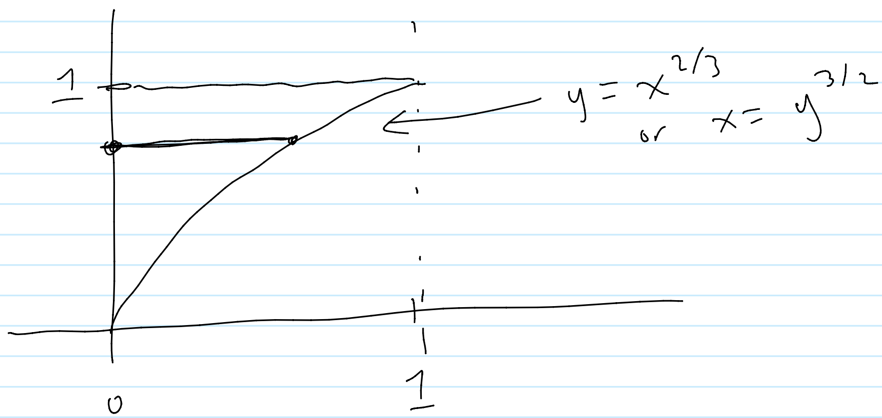
<u>Question</u>. change order of integration and evaluate:

Question: change order of integration and evaluate:

$$\int_0^1 \int_{x^{2/3}}^1 x e^{y^4} \, dy \, dx .$$

Answer:

domain: $x^{2/3} \leq y \leq 1, \quad 0 \leq x \leq 1$




rearranging gives $\quad 0 \leq y \leq 1, \quad 0 \leq x \leq y^{3/2}$.

so integral becomes

$$\int_0^1 \int_0^{y^{3/2}} x e^{y^4} \, dx \, dy = \int_0^1 \frac{x^2}{2} e^{y^4} \Big|_0^{y^{3/2}} dy$$

$$= \frac{1}{2} \int_0^1 y^3 e^{y^4} dy$$

$$= \frac{1}{8} e^{y^4} \Big|_0^1 = \frac{1}{8}(e - 1).$$

- Average of a function over a domain.

$$\bar{f} = \frac{1}{\text{Area}(D)} \iint_D f(x,y)\,dA.$$

alternatively, $\iint_D f(x,y)\,dA = \text{Area}(D) \cdot \bar{f}$.

- Mean value theorem. If $D$ closed, bounded, connected, exists $P \in D$ s.t.

$$\iint_D f(x,y)\,dA = f(P)\,\text{Area}(D).$$

we can use this to get approximations.

Example: let $D$ be rectangle $0 \le x \le 2$, $-\frac{1}{8} \le y \le \frac{1}{8}$ and $f(x,y) = \sqrt{x^3+1}$. Prove that

$$\iint_D f(x,y)\,dA \le 3/2.$$

sol$^n$: Note that $\text{Area}(D) = \frac{1}{2}$, and as $0 \le x \le 2$ and $f(x,y)$ increasing in $x$, $f(x,y) \le \sqrt{2^3+1} = 3$.

Hence by mean-value thm.

$$\begin{aligned} \iint_D f(x,y)\,dA &= f(P)\,\text{Area}(D) \\ &\le 3\,\text{Area}(D) \quad \text{as } f(P) \le 3 \text{ for all } P. \\ &= 3/2. \end{aligned}$$

Question: prove $\iint_D \frac{dA}{9+x^2+y^2} \le \pi$ where

$D$ = the disk $x^2+y^2 \le 9$

$\_\_\_\_\_\_\_\_ \quad \iint_D \frac{1}{9+x^2+y^2}$

$D$ is the disk $x^2+y^2 \leq 9$.

Answer: $f(x,y) = \dfrac{1}{9+x^2+y^2}$ is maximal when $x=y=0$

and $x^2+y^2 \leq 9$ is a disc of radius 3. Hence

$$\iint_D \frac{dA}{9+x^2+y^2} = f(P)\,\text{Area}(D)$$

$$\leq f(0,0)\,\pi 3^2$$

$$= \frac{1}{9}\pi \cdot 9 = \pi.$$

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