Week 10 Notes

Divergence Theorem:

From last week we know that it is

$$\iiint_W \operatorname{div}(\vec{F})\, dV = \iint_{\partial W} \vec{F} \cdot ds$$

where ∂W is a surface with outward facing normal.

How to interpret this?

If we picture $\vec{F}$ as the velocity field of a fluid, then the surface integral

$$\iint_{\partial W} \vec{F} \cdot ds$$

is the flow rate, ie the volume of liquid passing through ∂W per unit time.

We can think of $\operatorname{div} \vec{F}$ at a point as measuring the flow rate through a very small sphere around that point. Hence the divergence theorem says if we add all these up, it is equal to the flow rate through the boundary.

Question: Use the divergence theorem

to find $\iint_S F \cdot dS$ where

$F(x,y,z) = \langle xy^2, yz^2, zx^2 \rangle$ when S is the boundary of cylinder given by $x^2+y^2 \leq 4$ $0 \leq z \leq 3$.

Answer:

$$\text{div}(\vec{F}) = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(zx^2)$$

$$= y^2 + z^2 + x^2.$$

So by divergence:

$$\iint_S \vec{F} \cdot dS = \int_0^3 \iint_{\{x^2+y^2 \leq 4\}} x^2+y^2+z^2 \, dA \, dz$$

$$= \int_0^3 \int_0^2 \int_0^{2\pi} (r^2+z^2) r \, dr \, d\theta \, dz$$

$$= 2\pi \int_0^3 \int_0^2 r^3 + z^2 r \, dr \, dz$$

$$= 2\pi \int_0^3 \frac{r^4}{4} + \frac{z^2 r^2}{2} \Big|_0^2 dz$$

$$= 2\pi \int_0^3 4 + 2z^2 \, dz$$

$$= 2\pi \left(4z + \frac{2}{3}z^3\right)\Big|_0^3$$

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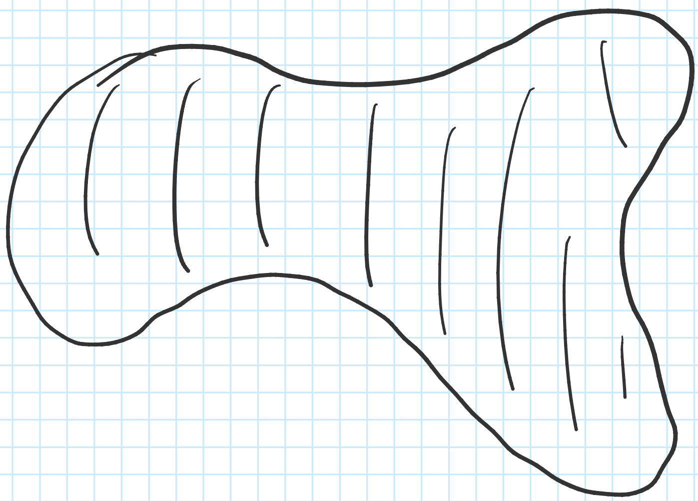
$$= 2\pi (12 + 18)$$

$$= 60\pi$$

Question Let S be the surface:

S: 

Find the volume enclosed by S given that

$$\iint_S \langle x, 2y, 2z \rangle \cdot dS = 50$$

Answer: Let W be the volume enclosed. Then $\partial W = S$. So by divergence theorem we have

$$\iiint_W \operatorname{div}(\langle x, 2y, 2z \rangle)\, dV = \iint_S \langle x, 2y, 2z \rangle \cdot dS = 50$$

but $\operatorname{div}(\langle x, 2y, 2z\rangle) = 1+2+2 = 5.$

Hence $5\iiint_W dV = 50$

$\therefore \text{Area}(W) = 10.$

Now A few questions that can use either fundamental theorem of line integrals / stokes or divergence.

Question: Show that the line integral of $\vec{F} = \langle x^2, y^2, z(x^2+y^2)\rangle$ around any closed curve C on the surface of the cone $z^2 = x^2+y^2$ is zero.

Answer (corrected since discussion)

We have $\nabla \times \vec{F} = \langle 2yz, -2xz, 0\rangle$

Now, we want to find $\vec{N}$.

Since $z^2 = x^2+y^2$, we have

$$2z\frac{\partial z}{\partial x} = 2x \Rightarrow \frac{\partial z}{\partial x} = x/z$$

$$2z \frac{\partial z}{\partial x} = 2y \Rightarrow \frac{\partial z}{\partial y} = y/z$$

Hence, given the parameterisation

$G(x,y) = (x, y, z(x,y))$

$G_x = (1, 0, x/z)$ $\quad G_y = (0, 1, y/z)$

$\vec{N} = G_x \times G_y = \left\langle -\frac{x}{z}, -\frac{y}{z}, 1 \right\rangle$

Hence $(\nabla \times \vec{F}) \cdot \vec{N} = 0$ and so it follows by Stokes theorem.

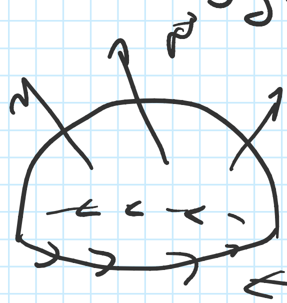
Question: Let S be the upper half of the ellipsoid $(z \geq 0)$ $\frac{x^2}{4} + y^2 + z^2 = 1$ with outward facing normal. Given $\vec{F} = \langle z^2, x+z, y^2 \rangle$.

what is $\iint_S \nabla \times \vec{F} \cdot dS = ?$

Answer:

By stokes theorem

$$\iint_S \nabla \times \vec{F} \cdot dS = \int_{\partial S} \vec{F} \cdot dr.$$



$\leftarrow$ induced orientation.

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We parameterize the boundary by $r(\theta) = (2\cos\theta, \sin\theta, 0)$

this has the correct orientation. $r'(\theta) = (-2\sin\theta, \cos\theta, 0)$

and so

$$\int_{\partial S} \vec{F} \cdot dr = \int_0^{2\pi} \langle 0, \tfrac{1}{2}\cos\theta, \sin^2\theta \rangle \cdot \langle -\tfrac{1}{2}\sin\theta, \cos\theta, 0 \rangle \, d\theta$$

$$= \int_0^{2\pi} 2\cos^2\theta \, d\theta$$

$$= 2\pi \quad \text{by symmetry.}$$

Question

Suppose for a region W we have that

$$\iint_{\partial W} \langle x + xy + z, x + 3y - \tfrac{1}{2}y^2, 4z \rangle \cdot dS = 16$$

What is $\mathrm{vol}(W) = ?$

Answer:

Divergence theorem:

$$\iiint_W \mathrm{div}(\vec{F}) \, dV = 16$$

$$\iiint_W 1 + y + 3 - y + 4 \, dV = 16$$

$$\int\int\int\int_W 8 \, dV = 16 \Rightarrow \mathrm{Vol}(W) = 2.$$