Recall Green's theorem: If $D \subset \mathbb{R}^2$ has boundary ∂D that is a simple closed curved curve oriented counterclockwise relative to D, then if $\mathbf{F} = \langle F_1, F_2 \rangle$ is a vector field with continuous partial derivatives defined on D then $\oint_{\partial D} \mathbf{F} \cdot dr = \iint_D \nabla \times F \, dA$, or to write it differenly $\oint_{\partial D} F_1 dx + F_2 dy = \iint_D \partial F_2/\partial x - \partial F_1/\partial y \, dA$.

1. Let $\mathbf{F} = \langle \sin x^2, xy \rangle$ and let C be the triangle with vertices $(0,0), (3,0), (3,2)$ oriented counterclockwise. What is $\oint_C \mathbf{F} \cdot dr$?

Solution: We could compute this directly by splitting the curve C into three straight lines, but since this week's worksheet is about Green's Theorem, let's use that tool. The curl of $\mathbf{F}$ is y, so by Green's Theorem we have

$$\iint_{\mathcal{D}} y \, \mathrm{d}A = \oint_C \mathbf{F} \cdot \, \mathrm{d}r$$

where $\mathcal{D}$ is the region bounded by C. We can now compute this double integral directly:

$$\oint_C \mathbf{F} \cdot \, \mathrm{d}r = \iint_{\mathcal{D}} y \, \mathrm{d}A = \int_0^3 \int_0^{2x/3} y \, \mathrm{d}y \, \mathrm{d}x = \frac{2}{9} \int_0^3 x^2 \, \mathrm{d}x = \frac{2}{9} \cdot \frac{27}{3} = 2.$$

2. Let $\mathbf{F} = \langle e^{y^2}, -e^{x^2} \rangle$ and let C be the oriented curve $r(t) = \langle \cos t, \sin t \rangle$ where $\pi/4 \leq t \leq 5\pi/4$. What is $\int_C \mathbf{F} \cdot dr$?

Just doing this directly is going to leave you with a tough (but not impossible!) integral. Talk with your group about how you might go about computing this. You can't use Green's theorem since C isn't a closed curve and you can't use path independence since the vector field isn't conservative. Can you find some other curve with the same end points as C where computing the line integral is easy and then use Green's theorem? At some point you might want to use some symmetry over the line $y = -x$, i.e. you might have a region where if (x, y) is in your region then $(-y, -x)$ is as well.

Solution: Let L be the line from $r(5\pi/4)$ to $r(\pi/4)$. This can be parametrized by

$$s(t) = \frac{1}{\sqrt{2}} \langle 2t - 1, 2t - 1 \rangle, 0 \leq t \leq 1$$

L and C together bound a half-disc $\mathcal{D}$, so Green's Theorem tells us that

$$\iint_{\mathcal{D}} \nabla \times \mathbf{F} \, \mathrm{d}A = \int_C \mathbf{F} \cdot \, \mathrm{d}r + \int_L \mathbf{F} \cdot \, \mathrm{d}r,$$

hence

$$\int_C \mathbf{F} \cdot \, \mathrm{d}r = \iint_{\mathcal{D}} \nabla \times \mathbf{F} \, \mathrm{d}A - \int_L \mathbf{F} \cdot \, \mathrm{d}r.$$

The curl of $\mathbf{F}$ is $(\nabla \times \mathbf{F})(x, y) = -(2xe^{x^2} + 2ye^{y^2})$. Notice that

$$(\nabla \times \mathbf{F})(-y, -x) = -(2(-y)e^{(-y)^2} + 2(-x)e^{(-x)^2}) = 2(xe^{x^2} + ye^{y^2}) = -(\nabla \times \mathbf{F})(x, y).$$

Since $\mathcal{D}$ is symmetric about the line $y = -x$, we get that

$$\iint_{\mathcal{D}} \nabla \times \mathbf{F} \, \mathrm{d}A = -\iint_{\mathcal{D}} \nabla \times \mathbf{F} \, \mathrm{d}A,$$

so

$$\iint_{\mathcal{D}} \nabla \times \mathbf{F} \, \mathrm{d}A = 0.$$

Next, we see that

$$\int_L \mathbf{F} \cdot \, \mathrm{d}r = \int_0^1 \mathbf{F}(s(t)) \cdot s'(t)\mathrm{d}t = \frac{1}{\sqrt{2}} \int_0^1 \mathbf{F}(s(t)) \cdot \langle 2, 2 \rangle \, \mathrm{d}t$$

Since the two components of $s(t)$ are equal, the two components of $\mathbf{F}(s(t))$ add to 0, thus $\mathbf{F}(s(t)) \cdot \langle 2, 2 \rangle = 0$, so we have

$$\int_L \mathbf{F} \cdot \, \mathrm{d}r = 0,$$

and thus

$$\int_C \mathbf{F} \cdot \, \mathrm{d}r = \iint_{\mathcal{D}} \nabla \times \mathbf{F} \, \mathrm{d}A - \int_L \mathbf{F} \cdot \, \mathrm{d}r = 0 - 0 = 0.$$

3. In the examples we've done so far we've used Green's theorem to compute a line integral by realizing that the line integral is equal to an integral over a region of the plane. We can also go in the other direction, Green's theorem can help us compute double integrals. In particular it can help us integrate 1, i.e. find the area of regions of the plane.

   (a) Find the area of the ellipse $(x/a)^2 + (y/b)^2 = 1$ using Green's theorem. **Hint:** The curl of $\langle 0, x \rangle$ is 1.

Solution: Let $\mathcal{E}$ be the ellipse. $\partial\mathcal{E}$ can be parametrized by

$$r(t) = \langle a\cos(t), b\sin(t) \rangle, 0 \le t \le 2\pi.$$

Now by Green's Theorem, we have

$$\text{Area of } \mathcal{E} = \iint_{\mathcal{E}} 1 \, \mathrm{d}A = \iint_{\mathcal{E}} (\nabla \times \langle 0, x \rangle) \, \mathrm{d}A = \oint_{\partial\mathcal{E}} \langle 0, x \rangle \cdot \, \mathrm{d}r$$
$$= \int_0^{2\pi} \langle 0, a\cos(t) \rangle \cdot \langle -a\sin(t), b\cos(t) \rangle \, \mathrm{d}t = \int_0^{2\pi} ab\cos^2(t)\mathrm{d}t = \frac{ab}{2} \int_0^{2\pi} (1 + \cos(2t))\mathrm{d}t$$
$$= ab\pi.$$

   (b) Find a parameteriation of the curve $x^{2/3} + y^{2/3} = 1$ and use Green's theorem to compute the area bounded by this curve. **Hint:** Let $x(t) = \cos^3 t$. $\int_0^{2\pi} \cos^4 t \, dt = 3\pi/4$ and $\int_0^{2\pi} \cos^6 t \, dt = 5\pi/8$.

Solution: We want the curve to be parametrized by $r(t) = \left\langle \cos^3(t), y(t) \right\rangle$ for some function y, so we need $\cos^2(t) + y(t)^{2/3} = 1$. Equivalently, we want $y(t)^{2/3} = 1 - \cos^2(t) = \sin^2(t)$, so it seems reasonable to choose $y(t) = \sin^3(t)$. Indeed we can check that

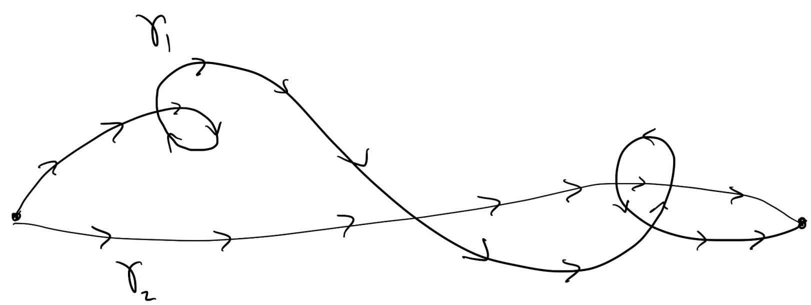
$$r(t) = \left\langle \cos^3(t), \sin^3(t) \right\rangle, 0 \le t \le 2\pi$$

parametrizes the desired curve. Now by Green's Theorem (using the same technique as in part (a)), the area bounded by the curve is just

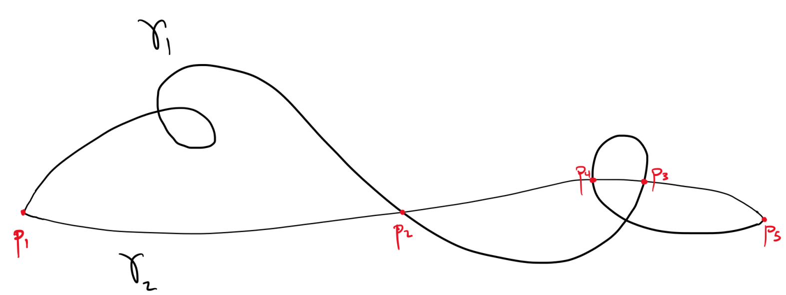
$$\int_0^{2\pi} \langle 0, \cos^3(t) \rangle \cdot \langle -3\cos^2(t)\sin(t), 3\sin^2(t)\cos(t) \rangle \, dt = 3\int_0^{2\pi} \sin^2(t)\cos^4(t)dt$$
$$= 3\int_0^{2\pi} (1-\cos^2(t))\cos^4(t)dt = 3\left(\int_0^{2\pi} \cos^4(t)dt - \int_0^{2\pi} \cos^6(t)dt\right)$$
$$= 3\left(\frac{3\pi}{4} - \frac{5\pi}{8}\right) = \frac{3\pi}{8}.$$

4. A few weeks ago we stated without proof that if a vector field with zero curl was defined on a simply connected region of the plane then line integrals over it are path independent. Use Green's theorem to show that this statement is true. Make sure that you consider how to deal with the case that two curves with the same beginning and end points might intersect at other points.

Solution: Let $\mathcal{D}$ be a simply connected domain over which a vector field $\mathbf{F}$ is defined, where $\nabla \times \mathbf{F} = 0$ on $\mathcal{D}$. Let $\gamma_1 : [0,1] \to \mathcal{D}$ and $\gamma_2 : [0,1] \to \mathcal{D}$ be two curves such that $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$.

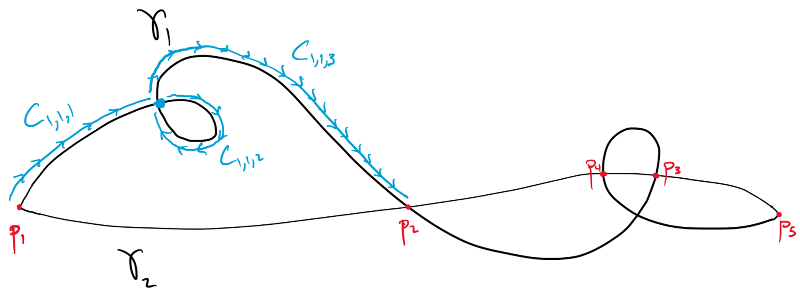


Let $p_1, \ldots, p_n$ be the distinct points of intersection of the two curves, listed in order of appearance on γ_1; that is, if $1 \le i < j \le n$ then there exist $t_1, t_2 \in [0,1]$ such that $t_1 < t_2$, $\gamma_1(t_1) = p_i$, and $\gamma_1(t_2) = p_j$. For each $1 \le i < n$, let $C_{1,i}$ be the portion of γ_1 between the points p_i and p_{i+1}; to be more precise, we want the portion of γ_1 between the first time it passes through p_i and the last time it passes through p_{i+1}. Likewise, let $C_{2,i}$ be the portion of γ_2 between the points p_i and p_{i+1}.



Now, for each $1 \le i < n$, the curve $C_{1,i}$ might intersect itself (perhaps multiple times) between its start and end points. If it does, let $r_i : [0,1] \to \mathcal{D}$ parametrize $C_{1,i}$. Let $t_{i,1}$ be the first value of t

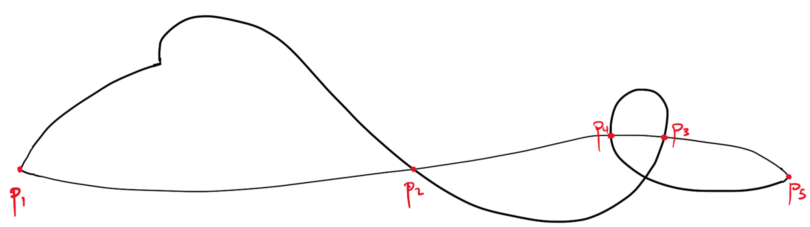
for whcih $r_i(t) = r_i(t')$ for some $t' > t$, and let $t_{i,2}, \ldots, t_{i,k_i}$ be the values of t (in increasing order) such that $r_i(t) = r_i(t')$ for some $t' < t$. For each $1 \leq j < k_i$, let $C_{1,i,j}$ be the portial of $C_{1,i}$ between the times $t_{i,j}$ and $t_{i,j+1}$.



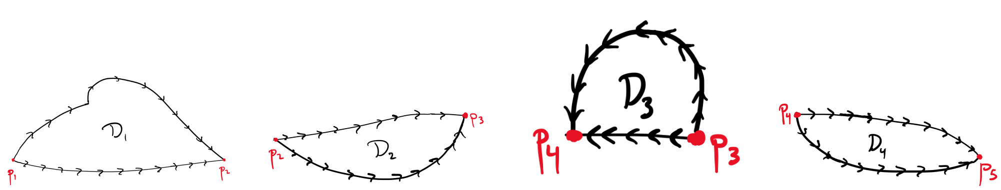
Now the curves $C_{1,i,j}$ are closed, and entirely contained in the simply connected region $\mathcal{D}$, so each bounds a closed region $\mathcal{D}_{i,j}$ over which the curl of $\mathbf{F}$ is zero. As a result, Green's theorem tells us that

$$\begin{aligned}\int_{C_{1,i}} \mathbf{F} \cdot \, \mathrm{d}r &= \int_0^{t_{i,1}} \mathbf{F}(r_i(t)) \cdot r_i'(t)\mathrm{d}t + \sum_{j=1}^{k_i-1} \oint_{C_{1,i,j}} \mathbf{F} \cdot \, \mathrm{d}r + \int_{t_{i,k_i}}^1 \mathbf{F}(r_i(t)) \cdot r_i'(t)\mathrm{d}t \\ &= \int_0^{t_{i,1}} \mathbf{F}(r_i(t)) \cdot r_i'(t)\mathrm{d}t + \sum_{j=1}^{k_i-1} \iint_{\mathcal{D}i,j} 0 \, \mathrm{d}A + \int_{t_{i,k_i}}^1 \mathbf{F}(r_i(t)) \cdot r_i'(t)\mathrm{d}t \\ &= \int_0^{t_{i,1}} \mathbf{F}(r_i(t)) \cdot r_i'(t)\mathrm{d}t + \int_{t_{i,k_i}}^1 \mathbf{F}(r_i(t)) \cdot r_i'(t)\mathrm{d}t.\end{aligned}$$

In other words, the integral over $C_{1,i}$ would be unchanged if we "skipped over" the part of the curve between the points where $C_{1,i}$ intersects itself. As a result, we may assume without loss of generality that $C_{1,i}$ *does not* intersect itself, since we will only care about the value of $\int_{C_{1,i}} \mathbf{F} \cdot \, \mathrm{d}r$. Likewise, we may assume that $C_{2,i}$ does not intersect itself for any $1 \leq i < n$.



Now, for each $1 \leq i < n$, the curve $C_{1,i} - C_{2,i}$ (or $C_{2,i} - C_{1,i}$) bounds a closed region $\mathcal{D}_i$ over which the curl of $\mathbf{F}$ is zero.



By Green's Theorem,

$$0 = \pm \iint_{\mathcal{D}_i} 0 \, \mathrm{d}A = \int_{C_{1,i} - C_{2,i}} \mathbf{F} \cdot \, \mathrm{d}r,$$

so

$$\int_{C_{1,i}} \mathbf{F} \cdot \, \mathrm{d}r = \int_{C_{2,i}} \mathbf{F} \cdot \, \mathrm{d}r.$$

Now

$$\int_{C_1} \mathbf{F} \cdot \, \mathrm{d}r = \sum_{i=1}^{n-1} \int_{C_{1,i}} \mathbf{F} \cdot \, \mathrm{d}r = \sum_{i=1}^{n_1} \int_{C_{2,i}} \mathbf{F} \cdot \, \mathrm{d}r = \int_{C_2} \mathbf{F} \cdot \, \mathrm{d}r,$$

as desired.