1. In this question we'll study the surface M with parameterization $G(u, v) = (x(u, v), y(u, v), z(u, v))$ where:

$$x(u, v) = 2\cos u + v\sin(u/2)\cos u$$
$$y(u, v) = 2\sin u + v\sin(u/2)\sin u$$
$$z(u, v) = v\cos(u/2)$$

where $0 \leq u \leq 2\pi$ and $-1 \leq v \leq 1$.

Think of $G(u, v)$ as the sum of two vector valued functions: $c(u) = \langle 2\cos u, 2\sin u, 0\rangle$ and $s(u, v) = v\langle\sin(u/2)\cos(u), \sin(u/2)\sin u, \cos(u/2)\rangle$.

(a) As u varies from 0 to 2π, what curve does c make?

(b) What is the length of the vector $s(u, v)$?

(c) What angle does $s(u, v)$ make with the z-axis? **Hint:** Recall that the angle θ between two vectors v, w is determined by $\cos\theta = \dfrac{v \cdot w}{||v||\,||w||}$.

(d) Observe that for a fixed u, $s(u, v)$ is a straight line. Note also that the projection of $s(u, v)$ into the xy-plane is parallel to $c(u)$.

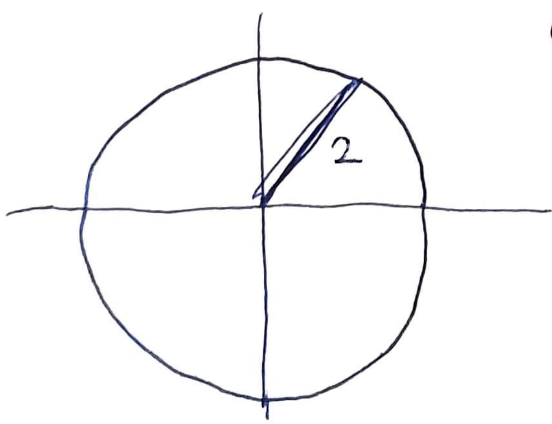
(e) What surface is M (the one that is parameteried by G)? It might help to plot the points where $v = \pm 1$ and u varies from 0 to 2π in multiples of $\pi/2$.

(f) Use this geogebra applet: `https://www.geogebra.org/m/BjV7cNwb` (or some other computer program) to visualize M. For each of questions a-d, relate what you found to properties of the surface M.

(g) Compute the normal vector to this surface. What it the normal vector when $u = 0$ and $v = 0$ and what it is when $u = 2\pi$ and $v = 0$? What does this tell you?

a) $c(u) = \langle 2\cos(u), 2\sin(u), 0\rangle, \quad 0 \leq u \leq 2\pi$

Circle with radius 2.



b) $$\|s(u,v)\| = \sqrt{v^2\sin^2\left(\tfrac{u}{2}\right)\cos^2(u) + v^2\sin^2\left(\tfrac{u}{2}\right)\sin^2(u) + v^2\cos^2\left(\tfrac{u}{2}\right)}$$

$$= \sqrt{v^2\sin^2\left(\tfrac{u}{2}\right) + v^2\cos^2\left(\tfrac{u}{2}\right)}$$

$$= v$$

(c) $\cos(\theta) = \dfrac{s(u,v)\cdot \hat{z}}{\|s(u,v)\| \, \|\hat{z}\|}$

$= \dfrac{v\cos(\frac{u}{2})}{v \cdot 1}$

$= \cos\left(\frac{u}{2}\right)$

$\Rightarrow \theta = \dfrac{u}{2}$

(d) $s(u,v) = v \underbrace{\left\langle \sin(\tfrac{u}{2})\cos(u), \sin(\tfrac{u}{2})\sin(u), \cos(\tfrac{u}{2}) \right\rangle}_{\text{Fixed vector}}$

$-1 \le v \le 1$, $s(u,v)$ traces out a line segment (if u fixed).

$s(u,v)\big|_{xy\text{-plane}} = v\left\langle \sin(\tfrac{u}{2})\cos(u), \sin(\tfrac{u}{2})\sin(u), 0\right\rangle$

$= v\sin(\tfrac{u}{2})\left\langle \cos(u), \sin(u), 0\right\rangle$

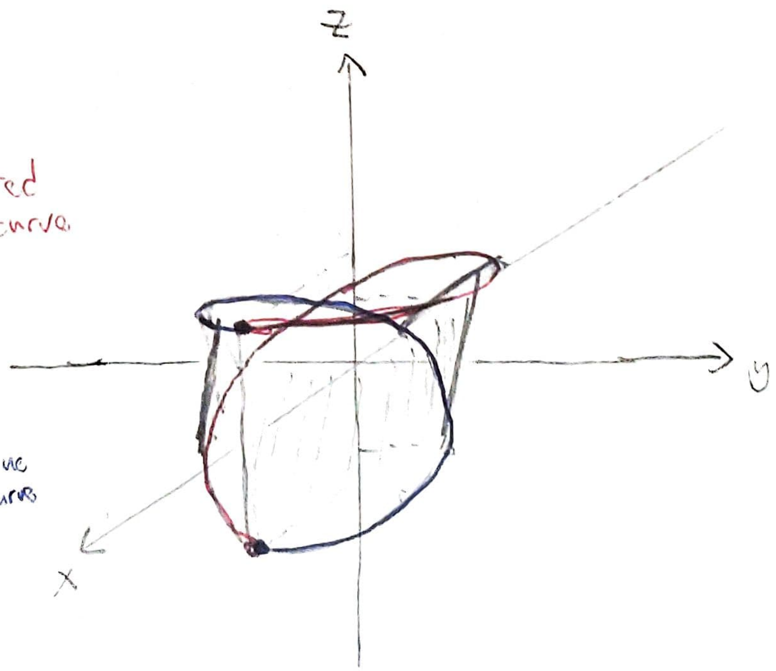
$= \dfrac{v\sin(\frac{u}{2})}{2} \cdot \vec{c}(u)$

$\Rightarrow s(u,v)\big|_{xy\text{-plane}} \parallel c(u)$

(e)

$$\left.\begin{aligned}
&= G(0,1) = \langle 2,0,1\rangle \\
&G(\tfrac{\pi}{2},1) = \langle 0, 2+\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}\rangle \\
&G(\pi,1) = \langle -3,0,0\rangle \\
&G(\tfrac{3\pi}{2},1) = \langle 0,-2-\tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}\rangle \\
&= G(2\pi,1) = \langle 2,0,-1\rangle
\end{aligned}\right\} \text{red curve}$$

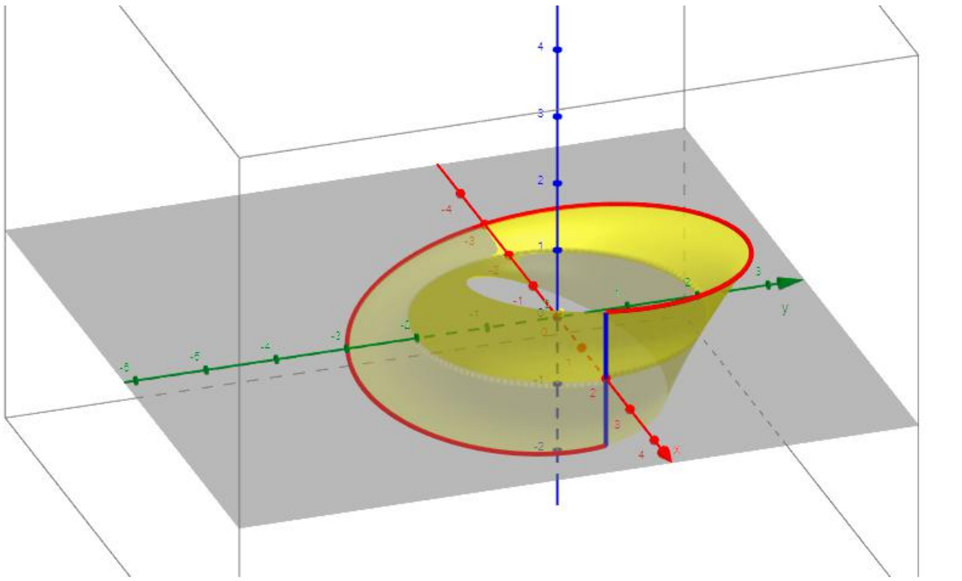
$$\left.\begin{aligned}
&G(0,-1) = \langle 2,0,-1\rangle \\
&G(\tfrac{\pi}{2},-1) = \langle 0, 2-\tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}\rangle \\
&G(\pi,-1) = \langle -1,0,0\rangle \\
&G(\tfrac{3\pi}{2},-1) = \langle 0, -2+\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}\rangle \\
&G(2\pi,-1) = \langle 2,0,1\rangle
\end{aligned}\right\} \text{blue curve}$$



$G(u,1)$, $G(u,-1)$ trace out the boundary curves of M as $0 \le u \le 2\pi$.

By part (d), we can connect the curves by line segments at each u, forming the surface.

f) The surface is a Möbius Strip:



Note that for fixed u, G(u,v) traces out a line segment, and for fixed v, G(u,v) spirals 2pi radians around the xy-plane

g) $$\frac{\partial G}{\partial u} = \left(-2\sin(u) + v\left(\tfrac{1}{2}\cos\left(\tfrac{u}{2}\right)\cos(u) - \sin\left(\tfrac{u}{2}\right)\sin(u)\right)\right)\hat{i}$$
$$+ \left(2\cos(u) + v\left(\tfrac{1}{2}\cos\left(\tfrac{u}{2}\right)\sin(u) + \sin\left(\tfrac{u}{2}\right)\cos(u)\right)\right)\hat{j}$$
$$+ \left(\frac{-v\sin\left(\frac{u}{2}\right)}{2}\right)\hat{k}$$

$$\frac{\partial G}{\partial v} = \sin\left(\tfrac{u}{2}\right)\cos(u)\hat{i} + \sin\left(\tfrac{u}{2}\right)\sin(u)\hat{j} + \cos\left(\tfrac{u}{2}\right)\hat{k}$$

$$\frac{\partial G}{\partial u}(0,0) = \langle 0, 2, 0\rangle$$

$$\frac{\partial G}{\partial v}(0,0) = \langle 0, 0, 1\rangle$$

$$N(0,0) = \langle 0,2,0\rangle \times \langle 0,0,1\rangle = \langle 2,0,0\rangle$$

$$\frac{\partial G}{\partial u}(2\pi,0) = \langle 0,2,0\rangle$$

$$\frac{\partial G}{\partial v}(2\pi,0) = \langle 0,0,-1\rangle$$

$$N(2\pi,0) = \langle 0,2,0\rangle \times \langle 0,0,-1\rangle = \langle -2,0,0\rangle$$

Note that $G(0,0) = G(2\pi,0) = \langle 2,0,0\rangle$

$N(0,0)$, $N(2\pi,0)$ point in opposite directions. After walking 2π radians along the strip, you are on the other side!

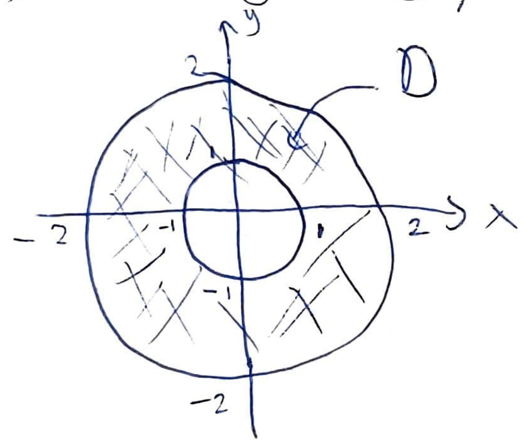
2. What is the area of the part of the surface $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$?

Since the surface is given in terms of the graph of a function $f(x,y)$, we can use the parameterization

$$G(x,y) = \langle x, y, y^2 - x^2 \rangle.$$

The region of integration in the parameter space is given by

$$\mathcal{D} = \{(x,y) : 1 \leq x^2+y^2 \leq 4\},$$ or geometrically by



To compute the surface area, we need to find the normal vector $N(x,y) = T_x \times T_y$:

$$T_x = \frac{\partial G}{\partial x} = \langle 1, 0, -2x \rangle$$

$$T_y = \frac{\partial G}{\partial y} = \langle 0, 1, 2y \rangle$$

$$N = \begin{vmatrix} i & j & k \\ 1 & 0 & -2x \\ 0 & 1 & 2y \end{vmatrix} = \langle 2x, -2y, 1 \rangle$$

$$\Rightarrow \|N\| = \sqrt{4x^2 + 4y^2 + 1}$$

$$SA = \iint_S dS = \iint_D \|N(x,y)\| \, dx\,dy$$

$$= \iint_D \sqrt{1+4x^2+4y^2} \, dx\,dy$$

We can compute this integral by converting to polar coordinates.

$1 \leq x^2+y^2 \leq 4 \implies 1 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi$

$dx\,dy = r\,dr\,d\theta$

$$SA = \int_0^{2\pi} \int_1^2 \sqrt{1+4r^2} \, r\,dr\,d\theta$$

$$= 2\pi \int_1^2 r\sqrt{1+4r^2}\,dr = 2\pi \int_{1+4(1)^2}^{1+4(2)^2} \sqrt{u} \, \frac{du}{8}$$

$$\boxed{\begin{aligned} u &= 1+4r^2 \\ du &= 8r\,dr \\ \implies r\,dr &= \frac{du}{8} \end{aligned}}$$

$$= \frac{\pi}{4} \int_5^{17} \sqrt{u}\,du$$

$$= \frac{\pi}{4} \left(\frac{2u^{3/2}}{3} \right) \Bigg|_5^{17}$$

$$= \frac{\pi}{6} \left(17^{3/2} - 5^{3/2} \right)$$

Note: You will arive at the same result by taking the parameterization

$G(r,\theta) = \langle r\cos\theta, r\sin\theta, r^2\sin^2\theta - r^2\cos^2\theta \rangle$,

$1 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$, but the computation of $\vec{N}$ will be a bit messier.

3. Consider the region bounded by two cylinders of radius one intersecting at right angles to each other, i.e. $\mathcal{B} = \{(x, y, z) : x^2 + y^2 \leq 1\} \cap \{(x, y, z) : x^2 + z^2 \leq 1\}$. This is called a *bicylinder*. See the picture above.

   (a) What is the surface area of $\mathcal{B}$? **Hint:** The bicylinder's surface comes in four parts, and each has the same surface area. To find the surface area of one part you can parameterize half of one of the cylinders by thinking of it as the graph of a function, and the domain for your parameters will be constrained by the other cylinder. Once you set up the integral computing surface area the integral is quick to do.

   Mathematicians of antiquity like Archimedes knew how to compute the volume of the bicylinder–it is much easier with calculus!

   (b) What is the volume of $\mathcal{B}$? For this, it probably again is a good idea to split $\mathcal{B}$ into four or even 8 pieces.

a) Let $z = \sqrt{1-x^2}$ be the graph that describes the upper half of the cylinder $x^2+z^2 \leq 1$.

Take the parameterization

$$G(x,y) = \langle x, y, \sqrt{1-x^2} \rangle.$$

On the domain $\mathcal{D} = \{(x,y): x^2+y^2 \leq 1\}$, G will trace out the upper ~~half~~ quarter of the bicylinder surface.

$$T_x = \frac{\partial G}{\partial x} = \left\langle 1, 0, -\frac{x}{\sqrt{1-x^2}} \right\rangle$$

$$T_y = \frac{\partial G}{\partial y} = \langle 0, 1, 0 \rangle$$

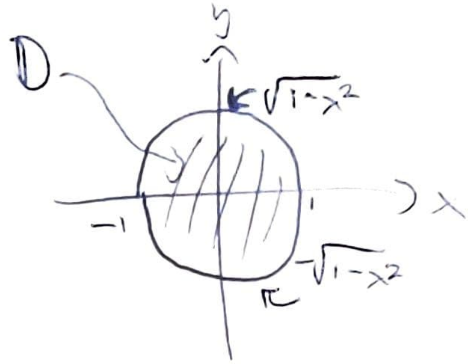
$$N = T_x \times T_y = \begin{vmatrix} i & j & k \\ 1 & 0 & \frac{-x}{\sqrt{1-x^2}} \\ 0 & 1 & 0 \end{vmatrix}$$

$$= \left\langle \frac{x}{\sqrt{1-x^2}}, 0, 1 \right\rangle$$

$$\|N\| = \sqrt{\frac{x^2}{1-x^2} + 1}$$

$$= \sqrt{\frac{x^2 + 1 - x^2}{1-x^2}}$$

$$= \frac{1}{\sqrt{1-x^2}}$$



$$\Rightarrow \frac{SA}{4} = \iint_S dS$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{dy\,dx}{\sqrt{1-x^2}}$$

$$= \int_{-1}^{1} \frac{2\sqrt{1-x^2}}{\sqrt{1-x^2}}\,dx$$

$$= \int_{-1}^{1} 2\,dx$$

$$= 4$$

$$\Rightarrow \boxed{SA_{\text{Bicylinder}} = 16}$$

b) Note that B is a z-simple region, whose projection onto the xy-plane is the circle $x^2+y^2 \le 1$.

$$\Rightarrow V = \iiint_B dV = \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \int_{z=-\sqrt{1-x^2}}^{z=\sqrt{1-x^2}} dz\,dy\,dx$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\sqrt{1-x^2}\,dy\,dx$$

$$= \int_{-1}^{1} 4(1-x^2)\,dx = 4x - \frac{4x^3}{3}\Big|_{-1}^{1} = \boxed{\frac{16}{3}}$$