1. Let $\mathbf{F}(x,y) = \langle y^2+1, 2xy-2\rangle$. Compute $\int_{\mathcal{C}} \mathbf{F}\cdot dr$ where $\mathcal{C}$ is:

   (a) The line segment from $(0,0)$ to $(1,1)$.

   > **Solution:**
   > We can parameterize this by: $r(t) = \langle t, t\rangle$ where $0 \le t \le 1$.
   > Then this integral is: $\int_0^1 \langle t^2+1, 2t^2-2\rangle \cdot \langle 1,1\rangle\, dt$. This is $\int_0^1 3t^2 - 1\, dt$ which is 0.

   (b) The path from $(0,0)$ to $(1,1)$ that first moves in a straight line to $(0,1)$ and then moves in a straight line to $(1,1)$.

   > **Solution:** The first path is $r(t) = \langle 0, t\rangle$ where $0 \le t \le 1$ and the second is $r(t) = \langle t, 1\rangle$ where $0 \le t \le 1$. So, this integral is $\int_0^1 \langle t^2+1, -2\rangle \cdot \langle 0,1\rangle\, dt + \int_0^1 \langle 2, 2t-2\rangle \cdot \langle 1,0\rangle\, dt = -2+2 = 0$.

   (c) Reconcile your answers with the fundamental theorem of conservative vector fields.

   > **Solution:** Since this vector field is conservative (a potential function is $xy^2 + x - 2y$) line integrals depend only on the endponts of a path, so we should get the same answer.

2. Let $f(x,y) = \sin x + x^2 y$ and let $\mathbf{F} = \nabla f$. Let $\mathcal{C}$ be the part of the parabola $y = x^2$ going from $(0,0)$ to (π, π^2).

   (a) Compute $\int_{\mathcal{C}} \mathbf{F}\cdot dr$ using the definition of vector line integrals.

   > **Solution:** Using the pamaterization of $\langle t, t^2\rangle, 0 \le t \le \pi$ this is $\int_0^\pi \langle \cos t + 2t^3, t^2\rangle \cdot \langle 1, 2t\rangle\, dt$ which is: $\int_0^\pi \cos t + 4t^3\, dt = \pi^4$

   (b) Compute $\int_{\mathcal{C}} \mathbf{F}\cdot dr$ using the fundamental theorem of conservative vector fields.

   > **Solution:** This is $f(\pi,\pi^2) - f(0,0) = \pi^4$.

3. Consider the vector field $\mathbf{F} = \langle 2xy^2z^2 + e^{x^2}, 2x^2yz^2 - e^{y^2}, 2x^2y^2z\rangle$. Let $\mathcal{C}$ be the part of the curve $\mathbf{r}(t) = \langle t, t^2, t\rangle$ $0 \le t \le 1$. Compute $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$.

   Before you embark on doing this problem, discuss with your group different ways that you might approach this problem.

   > **Solution:** Because of the e^{t^2} terms we can see that this integral is giong to be hard to do. We can compute the curl to see that this is conservative, but writing down a potential function involves integrating e^{x^2} and this doesn't have an elementary antiderivative.
   >
   > So, we can use path independence. This tells us that we can instead compute over the straight line from $(0,0,0)$ to $(1,1,1)$. We can tell this is going to be a good idea because the e^{x^2} and e^{y^2} pieces are going to cancel out.

This gives us $\int_0^1 2t^5 + e^{t^2} + 2t^2 - e^{t^2} + 2t^5 \, dt = 1$.

4. A vector field $\mathbf{F}$ has curl zero and is defined on all of $\mathbb{R}^2$ except for $(0,0)$ and $(2,0)$.

   (a) Show that if $\mathcal{C}$ is the circle of radius R with $R > 1$ centered at $(1,0)$ then $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is independent of R.

   Solution: If we take C_R of radius R and $C_{R'}$ of radius R', where $1 \leq R \leq R'$ then we can note that both circles contain both of the singularties. Since the curl of this vector field is zero it is conservative in the upper half plane. So, the line integral along the top half of $C_{R'}$ is equal to the line integral along the path that starts at $1 + R'$, travels in a straight line to $1 + R$, goes over the top half of C_R, and then travels in a straight line to $1 - R'$.

   Similarly the integral over the bottom half of $C_{R'}$ is equal to integrating over straight line segments and the bottom half of C_R. When we add together the two integrals involving C_R the straight lines cancel out, leaving us with the result.

   For more detail and some pictures, see notes from Monday or Friday of week 7.

   (b) Integrating $\mathbf{F}$ over a small circle centered at $(0,0)$ oriented counterclockwise gives 2 and integrating $\mathbf{F}$ over a small circle centered at $(2,0)$ oriented clockwise gives -1.

   What is the integral of $\mathbf{F}$ over the circle of radius R with $R > 1$ centered at $(1,0)$ oriented counterclockwise?

   Solution: The previous solution shows that line integrals over closed curves of curl zero vector fields depend only on the number of singularites enclosed by the curve and the curve's orientation. So, we could divide this circle of radius R in two by a vertical line through $(1,0)$ leaving us with two integrals, each over a curve containg one singularity. For the left hand circle, since it is oriented counterclockwise the integral over it will be equal to 2. For the right hand circle we get $+1$ because it is oriented counter clockwise and the integral over a clockwise circle gives 1.

   So, $\oint_{\mathcal{C}_{\mathcal{R}}} \mathbf{F} \cdot dr = 3$.