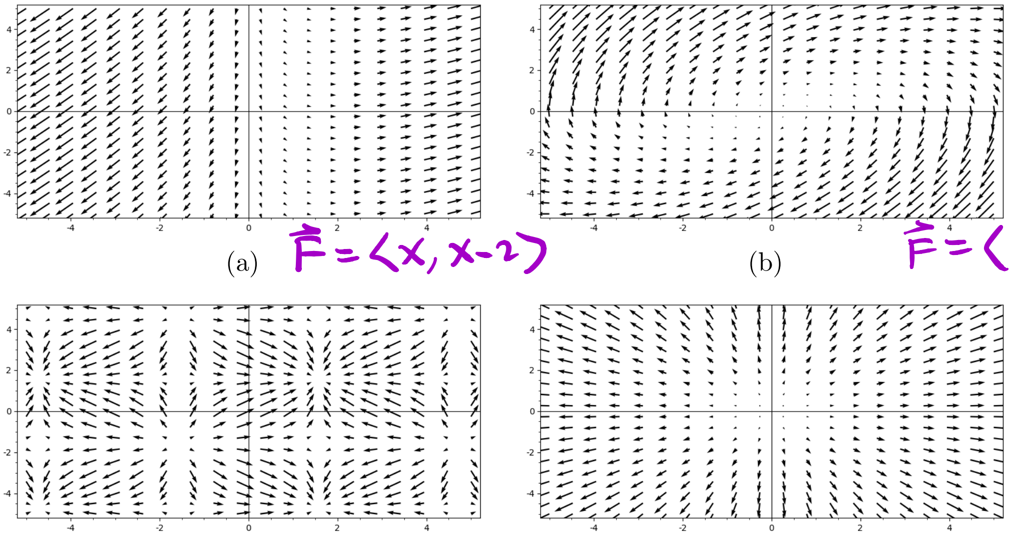

(a) $\vec{F} = \langle x, x-2 \rangle$

(b) $\vec{F} = \langle y, y-x \rangle$.

(c) $\vec{F} = \langle \cos x, \cos y \rangle$

(d) $\vec{F} = \langle x, y \rangle$

Figure 1

1. Match the vector field with its plot in the above figure. Talk with your group and relate characteristics of the plots to the functions.
   (a) $\mathbf{F}(x, y) = \langle x, y \rangle$
   (b) $\mathbf{F}(x, y) = \langle y, y - x \rangle$
   (c) $\mathbf{F}(x, y) = \langle \cos x, \cos y \rangle$
   (d) $\mathbf{F}(x, y) = \langle x, x - 2 \rangle$

2. Which of the above vector fields can you conclude are *not* conservative? For the others, can you find a potential function?

3. Show that if $\mathbf{F}(x, y, z)$ is a vector field with smooth component functions then the divergence of the curl of $\mathbf{F}$ is 0, i.e. $\nabla \cdot (\nabla \times F) = 0$. Give an explanation for why this is not surprising in light of what you know about $\cdot$ and $\times$ from 32a.

4. Try to determine whether or not the vector field $\mathbf{F}(x, y, z) = \langle y^2, 2xy + e^{3z}, 3ye^{3z} \rangle$ is conservative. If you think it is conservative, find a potential function.

5. Consider the vector field $\mathbf{F}(x, y) = \langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \rangle$.
   (a) Compute $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ where $\mathcal{C}$ is the circle of radius $r$ centered at the origin, oriented counterclockwise. This computation shows that $\mathbf{F}$ is *not* conservative. (We'll talk about this more in class on Wednesday)
   (b) Show that the curl of $\mathbf{F}$ is 0. Is this surprising in light of numbers 2 and 4 on this worksheet and that fact that $\mathbf{F}$ is not conservative (and what we've talked about in class, if you are doing this worksheet later in the week)? Give a resolution to the conundrum. **Hint:** What are the domains of all these vector fields?

# Question 2

We use the cross partials condition to test if a vector field is not conservative.

a) $\vec{F} = \langle x, y \rangle$

$$\partial F_1/\partial y = 0 = \partial F_2/\partial x.$$

So it passes and we try and find a potential function.

$$\nabla f = \langle x, y \rangle \Rightarrow f_x = x, \quad f_y = y.$$

Hence $f = \frac{x^2}{2} + \frac{y^2}{2}$ works.

b) $\vec{F} = \langle y, y-x \rangle$

$$\partial F_1/\partial y = 1 \neq -1 = \partial F_2/\partial x$$

Hence not conservative.

c) $\vec{F} = \langle \cos x, \cos y \rangle$

$\partial \vec{F}_1/\partial y = 0 = \partial F_2/\partial x.$

Now, $\nabla f = \langle \cos x, \cos y \rangle$

$f_x = \cos x, \quad f_y = \cos y.$

Hence $f = \sin x + \sin y$ works.

d) $\vec{F} = \langle x, x-2 \rangle$

$\partial \vec{F}_1/\partial y = 0 \neq 1 = \partial F_2/\partial x.$

Hence not conservative.

3. Show that if $\mathbf{F}(x, y, z)$ is a vector field with smooth component functions then the divergence of the curl of $\mathbf{F}$ is 0, i.e. $\nabla \cdot (\nabla \times F) = 0$. Give an explanation for why this is not surprising in light of what you know about $\cdot$ and $\times$ from 32a.

$\vec{F}(x,y,z) = \langle F_1, F_2, F_3 \rangle$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \langle \partial_y F_3 - \partial_z F_2, \partial_z F_1 - \partial_x F_3, \partial_x F_2 - \partial_y F_1 \rangle$$

$$\nabla \cdot (\nabla \times \vec{F}) = \partial_x \partial_y F_3 - \partial_x \partial_z F_2$$
$$+ \partial_y \partial_z F_1 - \partial_y \partial_x F_3$$
$$+ \partial_z \partial_x F_2 - \partial_z \partial_y F_1$$

$= 0$ by mixed partials derivative theorem.

This is unsurprising since in general $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$

$\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$ and so the dot product with $\vec{a}$ is zero.

4. Try to determine whether or not the vector field $\mathbf{F}(x, y, z) = \langle y^2, 2xy + e^{3z}, 3ye^{3z} \rangle$ is conservative. If you think it is conservative, find a potential function.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y^2 & 2xy + e^{3z} & 3ye^{3z} \end{vmatrix}$$

$$= \langle 3e^{3z} - 3e^{3z}, 0 - 0, 2y - 2y \rangle$$
$$= \vec{0}.$$

Since $\vec{F}$ defined on a simply connected domain we conclude $\vec{F}$ is conservative.

Now, we have $\vec{F} = \nabla f$ for some $f$.

Hence $f_x = y^2$ ... ①

$f_y = 2xy + 3e^z$ ... ②

$f_z = 3ye^{3z}$ ... ③

From ①: $f = \int f_x dx$

$f = xy^2 + C(y,z)$. ④

so $f_y = 2xy + C_y(y,z)$

equate with ②:

$2xy + C_y(y,z) = 2xy + 3e^z$

$\therefore\ C_y(y,z) = 3e^z$

so $C(y,z) = 3ye^z + D(z)$

Hence by (4): $f = xy^2 + 3ye^z + D(z)$

$f_z = 3ye^z + D_z(z)$

Equate with (3): $D_z(z) = 0$

Hence $D(z)$ is constant.

Let $D(z) = 0$.

then $f = xy^2 + 3ye^z$ is a potential function.

5. Consider the vector field $\mathbf{F}(x,y) = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$.

(a) Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where $C$ is the circle of radius $r$ centered at the origin, oriented counterclockwise. This computation shows that $\mathbf{F}$ is *not* conservative. (We'll talk about this more in class on Wednesday)

(b) Show that the curl of $\mathbf{F}$ is 0. Is this surprising in light of numbers 2 and 4 on this worksheet and that fact that $\mathbf{F}$ is not conservative (and what we've talked about in class, if you are doing this worksheet later in the week)? Give a resolution to the conundrum. **Hint:** What are the domains of all these vector fields?

(a) $\vec{r}(t) = \langle r\cos t, r\sin t \rangle$

$\vec{r}'(t) = \langle -r\sin t, r\cos t \rangle$

$\vec{F}(\vec{r}(t)) = \left\langle -\dfrac{\sin t}{r}, \dfrac{\cos t}{r} \right\rangle$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = \sin^2 t + \cos^2 t$

$= 1.$

Hence $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} 1 \cdot dt = 2\pi.$

(b) $\nabla \times \vec{F} = \left\langle 0, 0, \dfrac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \dfrac{x^2+y^2-2y^2}{(x^2+y^2)^2} \right\rangle$

$= \vec{0}.$

The domain of $\vec{F}$ isn't simply connected.