UCLA: Math 32B **Week 5 Worksheet** solutions **Spring 20**

1. (a) Find a change of coordinates $G : \mathbb{R}^3 \to \mathbb{R}^3$ so that if S^2 is the unit sphere then $G(S^2)$ is the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$.
   (b) Compute the volume ~~of~~ inside the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$.

2. If $W_0 \xrightarrow{G_1} W_1 \xrightarrow{G_2} W_2$ are two change of coordinates, what should the relationship be between $J(G_2 \circ G_1)$ and $J(G_1)$ and $J(G_2)$? Think geometrically in terms of what the Jacobian is measuring.

   Show that the relationship you posit is true when either ~~W_0 or W_1~~ G_1 or G_2 is a linear change of coordinates of the form $G(u, v, w) = (au, bv, cw)$. I suggest you do enough of the calculation to see what is happening, but to not necessarily compute the whole thing.

   What does this tell you about the first problem on the worksheet?

3. A spherical shell centered at the origin has an inner radius of 5 cm and an outer radius of 6 cm. The density of the material increases linearly with the distance from the center.

   At the inner surface the density is $12g/cm^3$ and at the outer surface the density is $14g/cm^3$.
   (a) Write the density in spherical coordinates.
   (b) Compute the mass of this spherical shell.
   (c) Since the density function is spherically symmetric the center of mass of the spherical shell is centered at the origin. Write down the integrals computing the center of mass and explain with minimal computations why they give that the center of mass is the origin.

4. (a) Find the center of mass of a solid ~~half~~ hemisphere with constant density . Because of the radial symmetry you know that the center of mass is located along the z-axis (if the hemipshere is sitting on the xy plane with one pole along the z-axis). Like in the previous question, write down the integrals computing the x and y coordinates of the center of mass and explain with minimal computations why they give that the center of mass is on the z-axis.
   (b) Compute the center of mass of the solid cone described by $\sqrt{x^2 + y^2} \le z \le 1$ if the cone has constant density.
   (c) If a solid $\mathcal{W}$ has center of mass (x_{CM}, y_{CM}, z_{CM}) does the plane $z = z_{CM}$ always divide $\mathcal{W}$ in to two regions with the same mass? If not, give a counterexample. What are some situations where the plane does divide $\mathcal{W}$ in to two regions of equal mass?

1. Let E be the ellipsoid.

a. $(x,y,z) \equiv G(u,v,w) = (au, bv, cw)$ takes the unit ball $B = \{(u,v,w) : u^2 + v^2 + w^2 \le 1\}$ to the ellipsoid. Since

$$G' = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix},$$

we have

b. $$\mathrm{Vol}(E) = \int_E dx\,dy\,dz = \int_{G(B)} dx\,dy\,dz = \int_B \underbrace{|\det G|}_{= abc} du\,dv\,dw = \frac{4}{3}\pi abc.$$

Note 1. We can take $a=b=c=r>0$, in which case we get the volume of a ball of radius r, as expected.

Note 2. The sphere $x^2+y^2+z^2=1$ is called S^2 because it's a 2-dimensional surface in R^3.

2. Let $y \in W_0$. By the chain rule, $(G_2 \circ G_1)'(y) = G_2'(G_1(y))\,G_1'(y)$. Thus,
$J(G_2 \circ G_1)(y) = J(G_2)(G_1(y))\,J(G_1)(y)$.
If, say, $G_2(u,v,w) = (au, bv, cw)$, we can do the computation explicitly.
For #1, if we do $(u,v,w) \xmapsto{G_1} (au, v, w) \xmapsto{G_2} (au, bv, w) \xmapsto{G_3} (au, bv, cw)$
then we get abc for the Jacobian of $G = G_3 \circ G_2 \circ G_1$, as we did above.

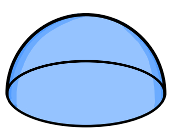
3a. We have $\delta(\rho,\theta,\phi) = \delta(\rho) = A\rho + B$ for some constants A, B.
We are given
$$12 = \delta(5) = 5A + B$$
$$14 = \delta(6) = 6A + B \quad \Rightarrow \quad A = 2,\ B = 2$$

b.
$$M = \int_0^{2\pi}\int_0^{\pi}\int_5^6 \delta(\rho)\,\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = 2\cdot 2\pi \int_0^{\pi} \sin\phi\, d\phi \cdot \int_5^6 (\rho+1)\rho^2\, d\rho$$
$$= 2\cdot 2\pi \cdot 2 \cdot \left(\frac{\rho^4}{4} - \frac{\rho^3}{3}\right)\Big|_5^6 = \frac{4754\pi}{3}$$

c.
$$x_{CM} = \frac{1}{M}\iiint_{5 \le \|(x,y,z)\| \le 6} x \cdot 2\left(\sqrt{x^2+y^2+z^2}+1\right) dx\, dy\, dz$$

odd in x, integrate over $\sqrt{5^2-(y^2+z^2)} \le |x| \le \sqrt{6^2-(y^2+z^2)}$, which is symmetric about 0. So this integral is 0.

sim. for y, z.

4a. 

Let's say the hemisphere has radius 1, density $\delta > 0$ (otherwise rescale)
$$x_{CM} = \frac{1}{M}\iiint_{\|(x,y,z)\| \le 1,\ z \ge 0} x \cdot \delta \; dx\, dy\, dz$$

this function is odd, and we integrate over $|x| \le \sqrt{1-(y^2+z^2)}$ for all y, z. So this integral is 0.

Similarly, $y_{CM} = 0$.

b. For the cone $\sqrt{x^2+y^2} \le z \le 1$, i.e. $r \le z \le 1$
$$M = \delta \operatorname{vol}(\text{cone}) = \delta \cdot \tfrac{1}{3}\pi r^2 h = \frac{\delta}{3}\pi$$
since the density is constant.

Hence,

$$z_{CM} = \frac{1}{M}\int_0^{2\pi}\int_0^1\int_r^1 z\,\delta r\,dz\,dr\,d\theta = \frac{\delta\pi}{4M} = \frac{\delta\pi}{4}\frac{3}{\delta\pi} = \frac{3}{4}.$$

c. Let's consider the hemisphere from part (a). We saw that $x_{CM} = y_{CM} = 0$, so the center of mass is $(0, 0, c)$ for some $c > 0$.

What is c? Note that $M = \delta\,\mathrm{vol}(\text{hemisphere}) = \frac{2\pi}{3}\delta$

$$c = z_{CM} = \frac{1}{M}\int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-z^2}} z\,\delta\,r\,dr\,dz\,d\theta = \frac{2\pi\delta}{M}\int_0^1 z\int_0^{\sqrt{1-z^2}} r\,dr\,dz$$

$$= \frac{\cancel{2}\pi\delta}{M}\int_0^1 \frac{z(1-z^2)}{\cancel{2}}\,dz = \frac{\pi\delta}{M}\cdot\frac{1}{4} = \frac{\pi\delta}{4}\cdot\frac{3}{2}\cdot\frac{1}{\pi\delta} = \frac{3}{8}$$

$$M_{\text{above } c} = \iiint_{\|(x,y,z)\|\le 1,\ z\ge c} \delta\,dx\,dy\,dz = \int_0^{2\pi}\int_c^1\int_0^{\sqrt{1-z^2}}\delta r\,dr\,dz\,d\theta$$

$$= 2\pi\delta\int_c^1\int_0^{\sqrt{1-z^2}} r\,dr\,dz = \pi\delta\int_c^1 (1-z^2)\,dz$$

$$= \pi\delta\left(\frac{2}{3} - c + \frac{1}{3}c^3\right) = \frac{475}{1536}\delta\pi \approx 0.309\,\delta\pi$$

Sim.

$$M_{\text{below } c} = \iiint_{\|(x,y,z)\|\le 1,\ 0\le z\le c} \delta\,dx\,dy\,dz = \int_0^{2\pi}\int_0^c\int_0^{\sqrt{1-z^2}}\delta r\,dr\,dz\,d\theta$$

$$= 2\pi\delta\int_0^c\int_0^{\sqrt{1-z^2}} r\,dr\,dz = \pi\delta\int_0^c (1-z^2)\,dz$$

$$= \pi\delta\left(c - \frac{1}{3}c^3\right) = \frac{183}{512}\delta\pi \approx 0.357\,\delta\pi$$

Note that $M = M_{\text{above } c} + M_{\text{below } c} = \frac{2}{3}\pi\delta = \mathrm{vol}(\text{hemisphere})\cdot\delta$, as desired.

For this example, there is more mass below the plane $z=c$.

Some examples where the mass is equal are a (full) sphere or a cube on top of the xy plane.