1. (a) Find a change of coordinates map G that takes the unit square $[0,1]\times[0,1]$ to the parallelogram with vertices $(0,0),(2,1),(1,2)(3,3)$.

Solution:

$$G(u,v) = (2u+v, u+2v)$$

(b) Find the Jacobian of G.

Solution:

$$J(G) = \det\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} = 3$$

(c) Find a change of coordinates map G' that takes the unit square $[0,1]\times[0,1]$ to the parallelogram with vertices $(2,1),(4,2),(3,3),(5,4)$.

Solution:

$$G'(u,v) = G(u,v) + (2,1) = (2u+v+2, u+2v+1)$$

(d) Find the Jacobian of G' and give a geometric explanation for the similarity between the Jacobian of G and that of G'.

Solution:

$$J(G') = \det\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} = 3$$

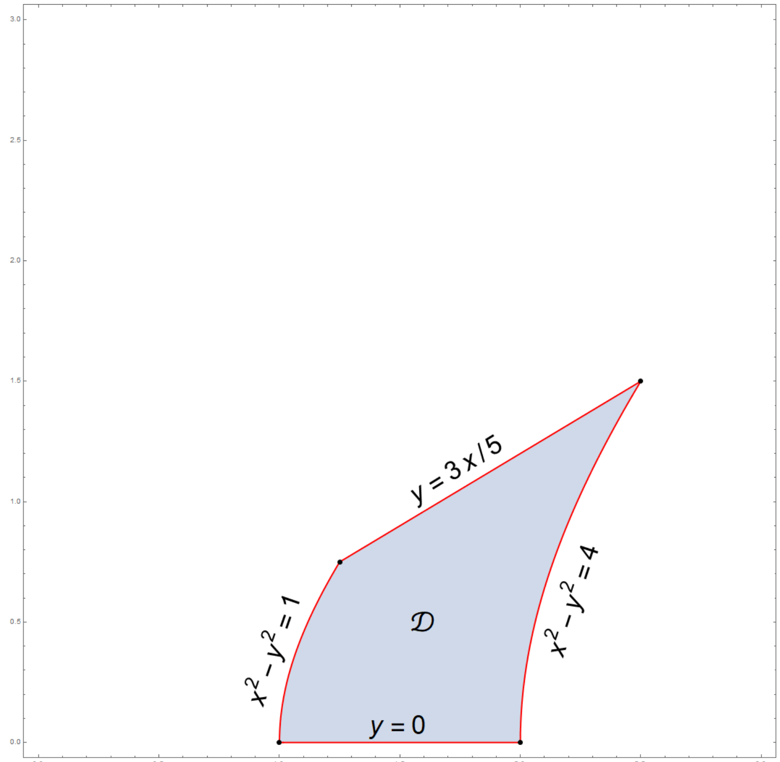
Since G and G' only differ by a translation, they scale areas by the same factor.

2. Consider the region $\mathcal{D}$ defined by $1 \le x^2 - y^2 \le 4$ and $0 \le y \le \frac{3x}{5}$. In this problem you'll set up an integral to compute $\iint_{\mathcal{D}} e^{x^2-y^2}\,\mathrm{d}A$.

Consider the change of coordinates $G(u,v) = \left(\frac{v}{2}+\frac{u}{2v}, \frac{v}{2}-\frac{u}{2v}\right)$. Recall from class that the inverse of this coordinate change is given by $G^{-1}(x,y) = (x^2-y^2, x+y)$

(a) Find a region $\mathcal{R}$ of the uv-plane so that $G:\mathcal{R}\to\mathcal{D}$ is a change of coordinates map (so G is onto and one-to-one on the interior of $\mathcal{R}$). **Hint:** Start by finding 4 curves in the uv-plane that map to the 4 curves forming the boundary of $\mathcal{D}$.

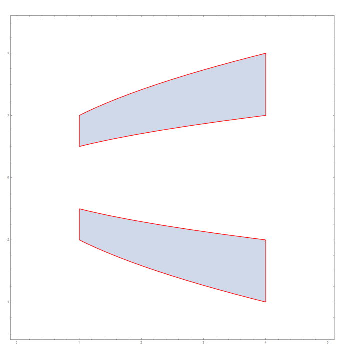
Solution: The boundary of $\mathcal{D}$ is defined by the four curves shown here:



Translating these equations into uv-coordinates, we get

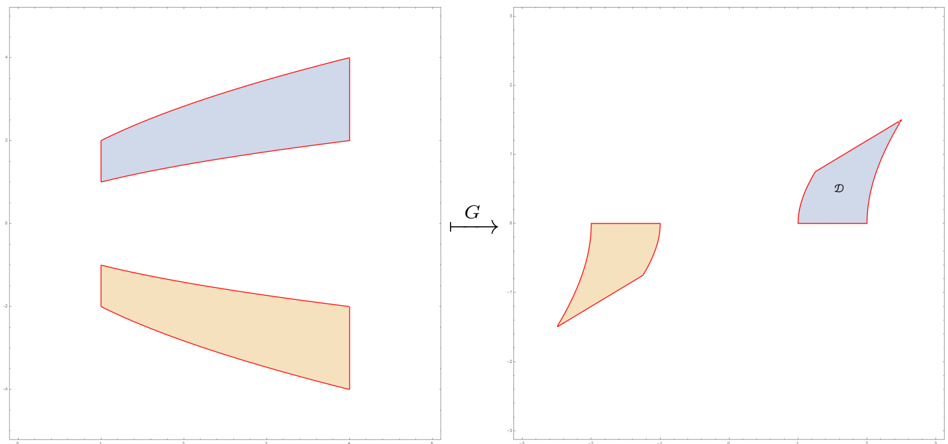
$$y = \frac{3x}{5} \iff \frac{v}{2} - \frac{u}{2v} = \frac{3}{5}\left(\frac{v}{2} + \frac{u}{2v}\right) \iff v^2 = 4u$$
$$x^2 - y^2 = 1 \iff u = 1$$
$$x^2 - y^2 = 4 \iff u = 4$$
$$y = 0 \iff v^2 = u$$

Thus, the region $\mathcal{R}$ should probably be defined by $u \le v^2 \le 4u$ and $1 \le u \le 4$. Let's plot this region in the uv-plane to make sure that's right:

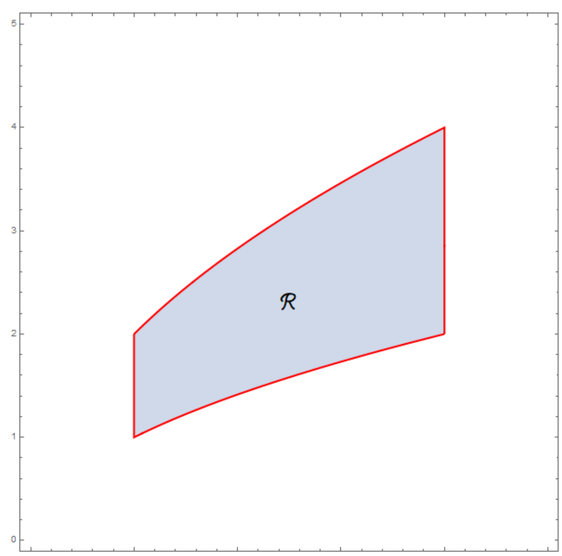


This seems a bit suspicious: we should expect the region $\mathcal{R}$ to be connected because G^{-1} is continuous! What went wrong?

Well, let's send this region we created over to the xy-plane via G:



Aha! It seems like we accidentally included a "mirror image" of $\mathcal{D}$. This mirror image also satisfies $1 \le x^2 - y^2 \le 4$, and in this new region y is still between 0 and $3x/5$, just backwards from the way we want: in this new region, we have $3x/5 \le y \le 0$. To fix this, we want to restrict ourselves to the part of the uv-plane where $v \ge 0$. An easy way to do this is to change the inequality $u \le v^2 \le 4u$ to the inequality $\sqrt{u} \le v \le 2\sqrt{u}$; this way we automatically exclude any points where v is negative, but otherwise keep everything else the same. Now the region defined by $1 \le u \le 4$ and $\sqrt{u} \le v \le 2\sqrt{u}$ really is the region $\mathcal{R}$ that we wanted:



(b) Give an iterated integral in uv-coordinatees to compute $\iint_{\mathcal{D}} e^{x^2-y^2}\, dA$ (No need to compute the actual integral, but it is an integral you can compute).

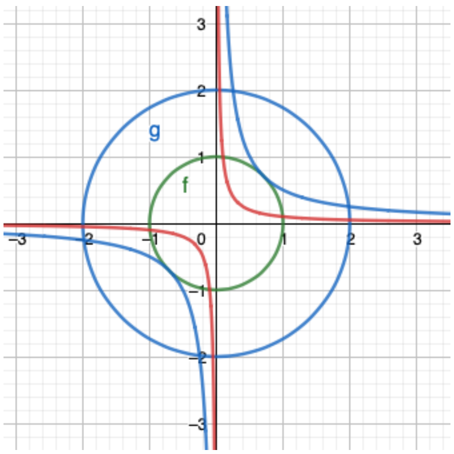
Solution: First, we need to compute the Jacobian of G. Perhaps the easiest way to do this is to compute the Jacobian of G^{-1} and invert that. We find that

$$J(G^{-1}) = \det \begin{bmatrix} 2x & -2y \\ 1 & 1 \end{bmatrix} = 2x + 2y = 2v,$$

so $J(G) = \dfrac{1}{2v}$. Now

$$\iint_{\mathcal{D}} e^{x^2-y^2}\,\mathrm{d}A = \iint_{\mathcal{R}} e^u |J(G)|\,\mathrm{d}A = \int_1^4 \int_{\sqrt{u}}^{2\sqrt{u}} \frac{e^u}{2v}\,\mathrm{d}v\,\mathrm{d}u = \frac{\log(2)(e^4-e)}{2}.$$

3. Consider the region of the part of the first quadrant $\mathcal{D}$ defined by $1 \leq x^2+y^2 \leq 4$ and $1/10 \leq xy \leq 1/2$ and $y \geq x$. There is a change of coordinates G that takes the rectangle $[1,4]\times[1/10,1/2]$ in the uv-plane to $\mathcal{D}$, and the *inverse* of this change of coordinates is given by $G^{-1}(x,y) = (x^2+y^2, xy)$.



(a) What is the absolute value of the Jacobian of G^{-1}? (It should be a function of x and y). Pay attention to signs!

Solution:
$$|J(G^{-1})| = \left|\det\begin{bmatrix} 2x & 2y \\ y & x\end{bmatrix}\right| = \left|2x^2-2y^2\right| = 2|x^2-y^2|$$

(b) Compute $\displaystyle\iint_{\mathcal{D}} y^2 - x^2\,\mathrm{d}A$

Solution: Since $y \geq x$ in the region $\mathcal{D}$, we have that $|J(G^{-1})| = 2|x^2-y^2| = 2(y^2-x^2)$ in the region $\mathcal{D}$. Thus,
$$\iint_{\mathcal{D}} y^2 - x^2\,\mathrm{d}A = \iint_{[1,4]\times[\frac{1}{10},\frac{1}{2}]} \frac{y^2-x^2}{2(y^2-x^2)}\,\mathrm{d}A = \frac{3}{5}$$

(c) Bonus problem: Note that the system of inequalites $1 \leq x^2+y^2 \leq 4$ and $1/10 \leq xy \leq 1/2$ defines *four* different regions of the plane. Each of these regions can be described by a change of coordinates G that takes the rectangle $[1,4]\times[1/10,1/2]$ in the uv-plane to the region where again *inverse* of this change of coordinates is given by $G^{-1}(x,y) = (x^2+y^2, xy)$, but for each of these regions G itself has a different formula. Find all 4 formula for G and say which of these four regions goes with which formula.

Solution: Let $G(x,y) = (u(x,y), v(x,y))$. We must always have

$$(u(x,y)^2 + v(x,y)^2, u(x,y)v(x,y)) = G^{-1}(u(x,y), v(x,y)) = G^{-1}(G(x,y)) = (x,y);$$

in other words, we have the equations

$$\begin{aligned} u^2 + v^2 &= x \\ uv &= y \end{aligned}$$

Now we see that $x + 2y = u^2 + 2uv + v^2 = (u+v)^2$, so $u + v = \pm\sqrt{x+2y}$. Likewise, $x - 2y = u^2 - 2uv + v^2 = (u-v)^2$, so $u - v = \pm\sqrt{x-2y}$. Knowing $u+v$ and $u-v$ allows us to solve for u and v, and there are two possibilities for each of these values, giving us four possibilities in total for G. These are:

$$\begin{aligned}
G(x,y) &= \left(\frac{\sqrt{x+2y}+\sqrt{x-2y}}{2}, \frac{\sqrt{x+2y}-\sqrt{x-2y}}{2}\right) \\
G(x,y) &= \left(\frac{-\sqrt{x+2y}+\sqrt{x-2y}}{2}, \frac{-\sqrt{x+2y}-\sqrt{x-2y}}{2}\right) \\
G(x,y) &= \left(\frac{\sqrt{x+2y}-\sqrt{x-2y}}{2}, \frac{\sqrt{x+2y}+\sqrt{x-2y}}{2}\right) \\
G(x,y) &= \left(\frac{-\sqrt{x+2y}-\sqrt{x-2y}}{2}, \frac{-\sqrt{x+2y}+\sqrt{x-2y}}{2}\right)
\end{aligned}$$

In the first equation for G, we see that both of the components are positive, and that the first component is greater than or equal to the second. Thus, this corresponds to the region in the first quadrant where $x \geq y$. Likewise, the second equation corresponds to the region in the third quadrant where $x \geq y$. The third equation corresponds to the region in the first quadrant where $x \leq y$, and the last equation corresponds to the region in the third quadrant where $x \leq y$.