1. We showed that the circle of radius $a$ has the concise description in polar coordinates as $r = a$.

   Find a description of the circles $(x-a)^2 + y^2 = a^2$ and $x^2 + (y-a)^2 = a^2$ in polar coordinates. (What if $a$ is negative?)

   *Solution*
   Before converting to polar coordinates, we can expand out the squares and simplify. The first circle satisfies the equation $x^2 - 2ax + a^2 + y^2 = a^2$, which implies that

   $$x^2 + y^2 = 2ax. \tag{1}$$

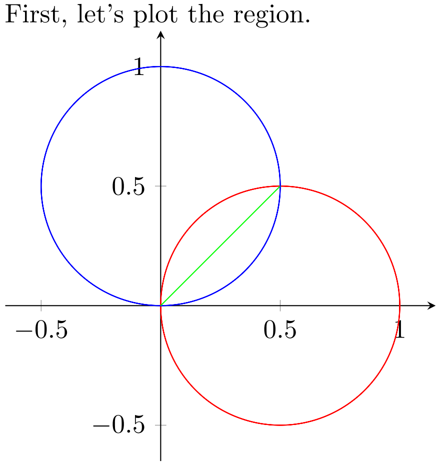
   Similarly, we find that the second circle satisfies the equation

   $$x^2 + y^2 = 2ay. \tag{2}$$

   Noting that $r^2 = x^2 + y^2$, and that $x = r\cos(\theta)$, $y = r\sin(\theta)$, we find that equation (1) becomes $r^2 = 2ar\cos(\theta)$, which implies that $r = 2a\cos(\theta)$, which is our polar description of the first circle. Similarly, (2) can be simplified to give us $r = 2a\sin(\theta)$. The sign of $a$ just tells us where the circles are centered. For example if $a$ is negative, the first circle will lie to the left and be tangent to the y-axis.

2. Compute the area inside the curves $r = \cos\theta$ and $r = \sin\theta$.

   *Solution*
   First, let's plot the region.



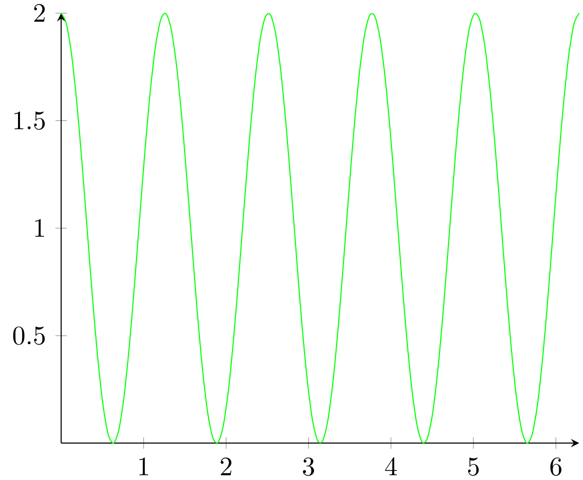

   We see that the curves intersect at $\theta = \pi/4$. The area of the region between the green segment and the blue curve $r = \sin(\theta)$ is precisely

   $$\begin{aligned}\int_0^{\frac{\pi}{4}} \int_0^{\sin(\theta)} r\,dr\,d\theta &= \int_0^{\frac{\pi}{4}} \frac{\sin^2(\theta)}{2} d\theta \\ &= \int_0^{\pi/4} \frac{1-\cos(2\theta)}{4} d\theta \\ &= \left[\frac{\theta}{4} - \frac{\sin(2\theta)}{8}\right]_0^{\pi/4} \\ &= \frac{\pi}{16} - \frac{1}{8}.\end{aligned}$$

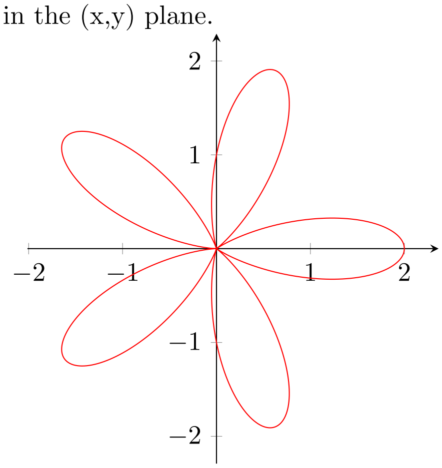
   By symmetry, the total area should be twice this, so $A = \dfrac{\pi}{8} - \dfrac{1}{4}$

3. (a) Plot the curve $r = 1 + cos(5\theta)$ (it should look something like a flower).

*Solution*
First, it may be helpful to graph this function in the $(\theta, r)$ plane:




We see that $r \geq 0$, and $r = 0$ when $\theta = \dfrac{\pi}{5}, \dfrac{3\pi}{5}, \pi, \dfrac{7\pi}{5}, \dfrac{9\pi}{5}$. Each petal will lie between these angles where $r = 0$. The petal on the x-axis is half drawn as we trace the curve from 0 to $\pi/5$, and is completed when we trace the curve from $9\pi/5$ to $2\pi$. Using this information, we can plot the curve in the (x,y) plane.




(b) Find the area of one "petal" of the curve $r = 1 + \cos 5\theta$. You will probably need to use the double angle formula.

*Solution*
Let's calculate the area of the petal lying on the x-axis (by symmetry all petals should have the same area). One way we could do this is to find the area of the half of the petal lying above the

x-axis, and then double the result. Then

$$\begin{aligned}
\text{Area of petal} &= 2\int_0^{\frac{\pi}{5}}\int_0^{1+\cos(5\theta)} r\,dr\,d\theta \\
&= 2\int_0^{\frac{\pi}{5}} \frac{(1+\cos(5\theta))^2}{2}\,d\theta \\
&= \int_0^{\frac{\pi}{5}} \left[1+2\cos(5\theta)+\cos^2(5\theta)\right] d\theta \\
&= \int_0^{\frac{\pi}{5}} \left[1+2\cos(5\theta)+\frac{1+\cos(10\theta)}{2}\right] d\theta \\
&= \int_0^{\frac{\pi}{5}} \left[\frac{3}{2}+2\cos(5\theta)+\frac{\cos(10\theta)}{2}\right] d\theta \\
&= \left[\frac{3\theta}{2}+\frac{2\sin(5\theta)}{5}+\frac{\sin(10\theta)}{20}\right]_0^{\pi/5} \\
&= \frac{3\pi}{10}.
\end{aligned}$$

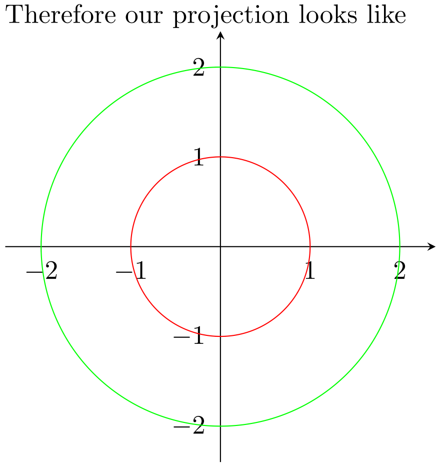
4. If a solid in a region $\mathcal{W} \subset \mathbb{R}^3$ has density given by $\partial(x,y,z)$ then its mass is given by $\iiint_{\mathcal{W}} \partial(x,y,z)\,\mathrm{d}V$.

   Suppose that we are measuring in meters and consider a solid that is above the plane $z=0$, below the paraboloid $z=4-(x^2+y^2)$, and outside the cylinder $x^2+y^2=1$. Suppose that the density of this solid is inversely proportional to the distance from the $z$-axis and that the density of this solid along the boundary where the paraboloid hits the $xy$-plane is $1/2\ kg/m^3$.

   Compute the mass of this solid.

   *Solution*

   Since the region $W$ is $z$-simple, bounded below by $z=0$ and above by $z=4-x^2-y^2$, we need to find the projection of $W$ onto the $x-y$ plane. The paraboloid and the z-axis intersect when $z=0$, which implies that $0=4-(x^2+y^2)$. Therefor the circle $x^2+y^2=4$ forms the outer boundary of the projection. The inner boundary is formed by the cylinder, which projects the circle $x^2+y^2=1$. Therefore our projection looks like



and the region on integration is between the red and green circles.

Next, let's figure out a formula for $\delta(x, y, z)$. We are told that $\delta$ is inversely proportional to the distance from the z-axis, so

$$\delta(r) = \frac{C}{r}$$

for some constant $C$. We also know that the density is $\frac{1}{2}$ kg$/m^2$ when the paraboloid hits the xy-plane, which means that $\delta(2) = \frac{1}{2}$. This suggests that $C = 1$, so $\delta(r) = \frac{1}{r}$. Finally, we'll set up the integral, and use cylindrical coordinates.

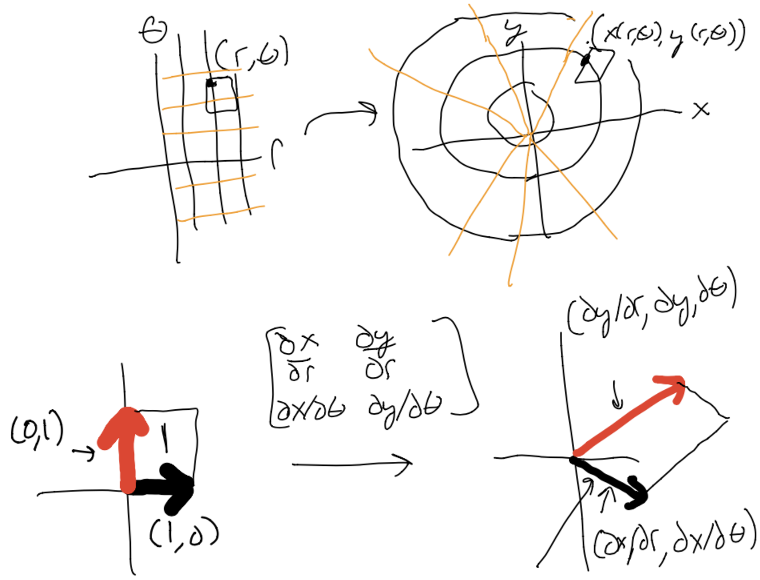
$$\begin{aligned} M &= \int_0^{2\pi} \int_1^2 \int_0^{4-r^2} \frac{1}{r} r dz dr d\theta \\ &= \int_0^{2\pi} \int_1^2 4 - r^2 dr d\theta \\ &= \int_0^{2\pi} \left[4r - \frac{r^3}{3}\right]_1^2 d\theta \\ &= \int_0^{2\pi} \frac{5}{3} d\theta \\ &= \frac{10\pi}{3} \end{aligned}$$

5. In this problem you will find the area of the ellipse $(x/a)^2 + (y/b)^2 = 1$. We'll use a distorted version of polar coordinates. We'll measure points in the plane by the angle $\theta$ the line from the origin to the point makes with the $x$-axis and the value of $r$ for which the point lies on the ellipse $\left(\frac{x}{ar}\right)^2 + \left(\frac{y}{br}\right)^2 = 1$.

   (a) Using these coordinates what point in the $xy$-plane does the value $(r, \theta)$ correspond to?
   See picture below:

   (b) Describe the ellipse $(x/a)^2 + (y/b)^2 = 1$ in these new "distorted polar coordinates". See picture below:

   (c) What is the distortion factor for area with these coordinates?
   For this, consider the map $(r, \theta) \mapsto (x(r, \theta), y(r, \theta))$ from the first part of the question. Differentiating this gives the matrix $\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{bmatrix}$, and the area distortion factor at $(r, \theta)$ is the determinant of this matrix, the quantity $\frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial y}{\partial r}\frac{\partial x}{\partial \theta}$. See the picture below.




the area of the square $[0,1]\times[0,1]$ under the this map is $\partial x/\partial r \cdot \partial y/\partial\theta - \frac{\partial y}{\partial r}\cdot\frac{\partial x}{\partial \theta}$

(d) What is the area enclosed by this ellipse?

*Solution*

a Choose $x(r,\theta) = ar\cos(\theta)$ and $y(r,\theta) = br\sin(\theta)$. We can verify that $(x(r,\theta), y(r,\theta))$ lies on the ellipse by computing

$$\left(\frac{x(r,\theta)}{ar}\right)^2 + \left(\frac{y(r,\theta)}{br}\right)^2 = \cos^2(\theta) + \sin^2(\theta)$$
$$= 1,$$

Note: It turns out that $(ar\cos(\theta), br\sin(\theta))$ is not the point of intersection between the ray making angle *theta* with the x-axis and the ellipse $\left(\frac{x}{ar}\right)^2 + \left(\frac{y}{br}\right)^2 = 1$ as the problem states. The actual point of intersection is

$$(x,y) = \left(\frac{abr\cos(\theta)}{\sqrt{a^2\sin^2(\theta) + b^2\cos^2(\theta)}}, \frac{abr\sin(\theta)}{\sqrt{a^2\sin^2(\theta) + b^2\cos^2(\theta)}}\right),$$

which can be computed by substituting $y = \tan(\theta)x$ into the equation for $r-$scaled ellipse. This transformation has a very messy Jacobian, and won't help us compute the area of the ellipse $(x/a)^2 + (y/b)^2 = 1$. Instead, we will just define the map by $(x(r,\theta), y(r,\theta)) = (ar\cos(\theta), br\sin(\theta))$, and ignore the geometric definition stated in the problem. All that matters is that our map transforms rectangles in the $r\theta$-plane (with height $2\pi$) to ellipses in the $xy-plane$, which we will see is the case.

b The ellipse $(x/a)^2 + (y/b)^2 = 1$ corresponds to the line segment $r = 1$, $0 \le \theta \le 2\pi$ in the $r\theta$ plane. This implies that if we mapped the whole region bounded by this ellipse to the $r\theta$ plane, we would form the rectangle $[0,1]\times[0,2\pi]$, see below picture.

c In part (a) we found that

$$x(r,\theta) = ar\cos(\theta)$$
$$y(r,\theta) = br\sin(\theta)$$

Then the distortion factor is

$$\begin{aligned}\frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial y}{\partial r}\frac{\partial x}{\partial \theta} &= (a\cos(\theta))(br\cos(\theta)) - (b\sin(\theta))(-ar\sin(\theta)) \\ &= abr\cos^2(\theta) + abr\sin^2(\theta) \\ &= abr.\end{aligned}$$

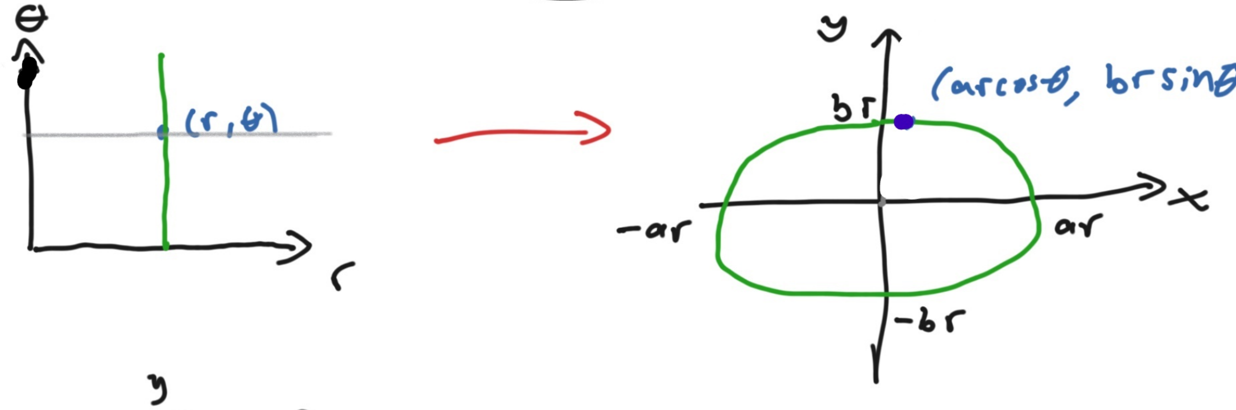
d Call

$$D = \left\{(x,y) \in \mathbb{R}^2 : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1\right\},$$

which represents the interior of the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ in the xy-plane. In part (b), we found that $D$ corresponds to the rectangle $D_0 = [0,1] \times [0,2\pi]$ in the $r\theta$ - plane. By the change of variables formula,

$$\begin{aligned}\text{Area of ellipse} = \int\int_D 1dxdy &= \int\int_{D_0} 1\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| drd\theta \\ &= \int_0^{2\pi}\int_0^1 abrdrd\theta \\ &= \int_0^{2\pi} ab\left[\frac{r^2}{2}\right]_0^1 d\theta \\ &= \int_0^{2\pi} \frac{ab}{2}d\theta \\ &= \pi ab.\end{aligned}$$

a)




b)