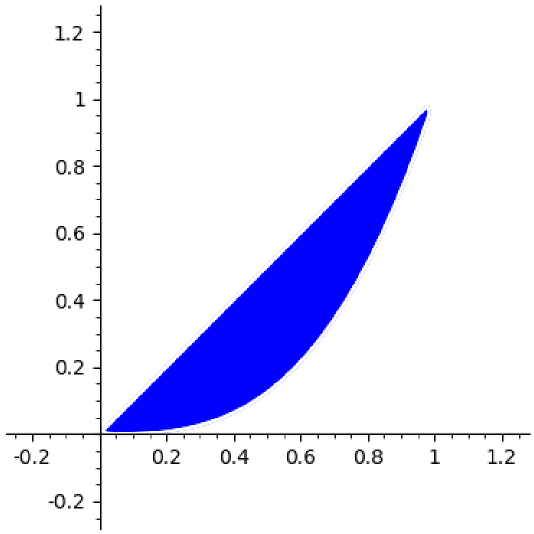
1. Let $\mathcal{D}$ be the region in the first quadrant of the plane bounded by the curves $y = x$ and $y = x^3$. Write the integral $\int\int_{\mathcal{D}} f(x, y)\, \mathrm{d}A$ in the two possible orders.

Solution: First, let's draw a picture.



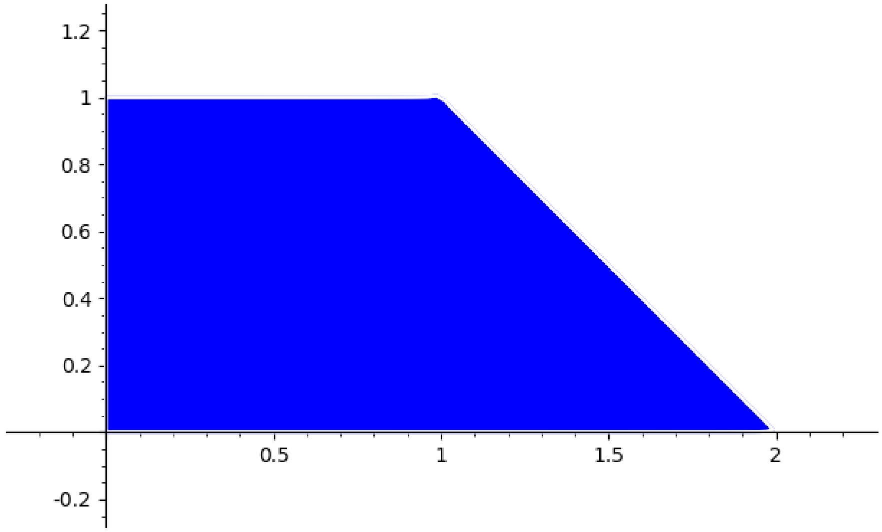
Note that the two curves intersect at $(0, 0)$ and at $(1, 1)$. We find the x-values by solving $x = x^3$ and then the y-values by plugging out solutions into the equation $y = x$.

So, when we describe this region as vertically simple it is: $0 \leq x \leq 1$ and $x^3 \leq y \leq x$. Note that for $x \in [0, 1]$ we have that $x^3 \leq x$. When we describe it as horizontally simple we have $0 \leq y \leq 1$ and $y \leq x \leq y^{1/3}$. We found the curves $x = y$ and $x = y^{1/3}$ by solving $y = x$ and $y = x^3$ for x. Note that for $y \in [0, 1]$ we have that $y \leq y^{1/3}$.

So, our twin integrals are $\int_0^1 \int_{x^3}^{x} f(x, y)\, \mathrm{d}y\, \mathrm{d}x$ and $\int_0^1 \int_{y}^{y^{1/3}} f(x, y)\, \mathrm{d}x\, \mathrm{d}y$.

2. Let $\mathcal{D}$ be the region in the plane bounded by the lines $y = 0$, $x = 0$, $y = 1$, and $y = -x + 2$. Write the double integral $\int\int_{\mathcal{D}} f(x, y)\, \mathrm{d}A$ as an iterated integral in both possible orders. Which is probably going to be less work to compute?

Solution: First, let's draw a picture.



This region is both vertically and horizontally simple, but it we describe it as vertically simple we need to use a piecewise function for our upper bound.

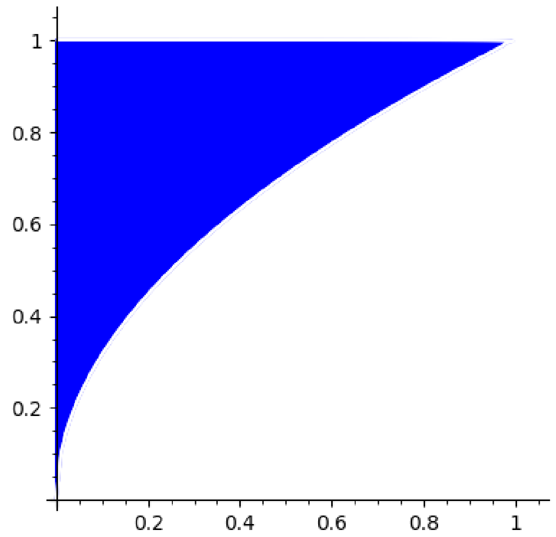
If we desribe it as vertically simple, we have that $0 \leq x \leq 2$ and $0 \leq y \leq g(x)$, where $g(x) = \begin{cases} 1 & x \leq 1 \\ -x+2 & x > 1 \end{cases}$.

If we describe it as horizontally simple we have $0 \leq y \leq 1$ and $0 \leq x \leq 2 - y$.

So, our two integrals are $\int_0^1 \int_0^{g(x)} f(x,y)\ \mathrm{d}y\ \mathrm{d}x = \int_0^1 \int_0^1 f(x,y)\ \mathrm{d}y\ \mathrm{d}x + \int_1^2 \int_0^{2-x} f(x,y)\ \mathrm{d}y\ \mathrm{d}x$ and $\int_0^1 \int_0^{2-y} f(x,y)\ \mathrm{d}x\ \mathrm{d}y$. Probably the last integral is the easiest since we don't have to split it in two to compute it.

3. Compute the double integral $\int\int_{\mathcal{D}} \sqrt{y^3+1}\ \mathrm{d}A$ where $\mathcal{D}$ is the region of the first quadrant bounded by $y = 1$ and $y = \sqrt{x}$. Try to compute it in both orders– is one way easier than the other?

Solution: First, let's draw a picture.



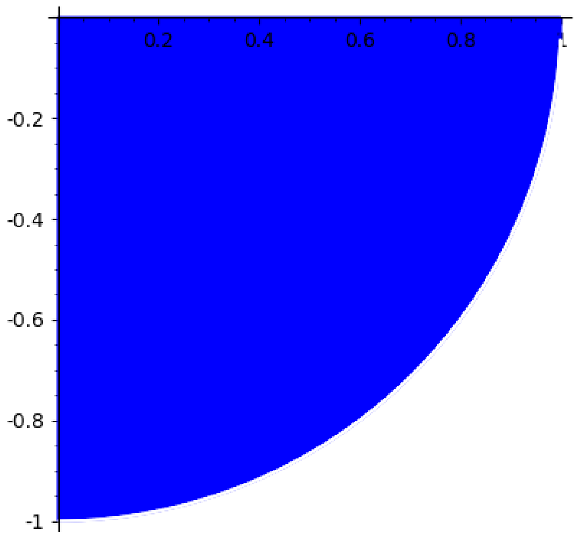
If we describe this region as vertically simple we have that $0 \leq x \leq 1$ and $\sqrt{x} \leq y \leq 1$. We got this by noting that the curve $y = \sqrt{x}$ intersects the curves $y = 1$ at $(1, 1)$, and noting that $\sqrt{x} \leq 1$ for $0 \leq x \leq 1$. If we describe this as horizontally simple thant we have that $0 \leq y \leq 1$ and $0 \leq x \leq y^2$. We found this bound by solving $y = \sqrt{x}$ for x.

So, our two integrals are $\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3+1}\, \mathrm{d}y\, \mathrm{d}x$ and $\int_0^1 \int_0^{y^2} \sqrt{y^3+1}\, \mathrm{d}x\, \mathrm{d}y$. We won't get anywhere trying to compute the first one since $\sqrt{y^3+1}$ doesn't have a nice antiderivative.

However, if we look at the second integral we see that there is a lovely y^2 term in the bounds that will come to our rescue so we can do a u-substitution. So, for the second integral after computing the inner integral we have $\int_0^1 y^2 \sqrt{y^3+1}\, \mathrm{d}y$. Doing the u-subtitution of $u = y^3 + 1$, $\mathrm{d}u = 3y^2\, \mathrm{d}y$ we have $1/3 \int_0^1 \int \sqrt{u} \mathrm{d}u = 2/9(2\sqrt{2} - 1)$.

4. Compute the integral $\int_0^1 \int_{-\sqrt{1-x^2}}^0 2x \cos(y - y^3/3)\, \mathrm{d}y\, \mathrm{d}x$.

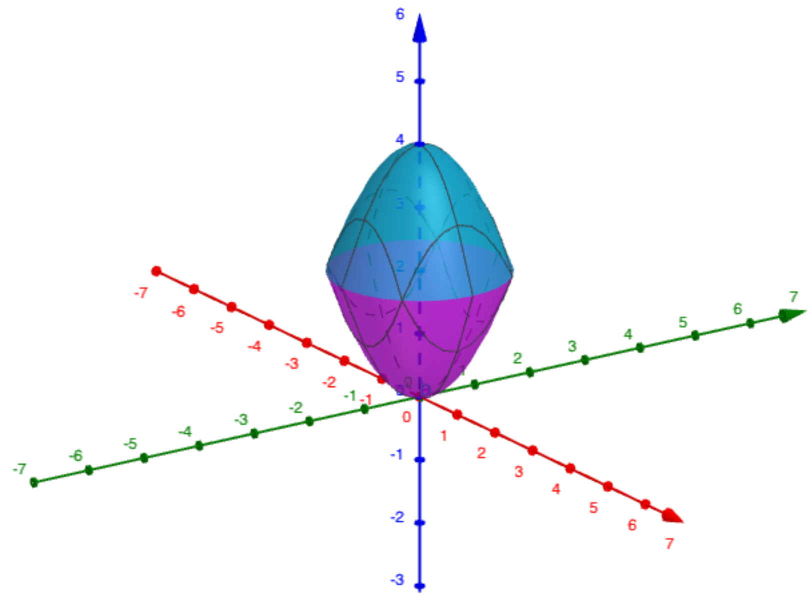
Solution: First, we draw a picture.



We immediately see that we have the same issue as the last problem when we try to integrate this as a vertically simple region: the function $\cos(y - y^3/3)$ has no elementary antiderivative. So, we change the order of integration. Note that we are integrating over the part of the unit disc in the fourth quadrant. When we change the order we get $\int_{-1}^{0}\int_{0}^{\sqrt{1-y^2}} 2x\cos(y-y^3/3)\ \mathrm{d}x\ \mathrm{d}y$. Integrating once gives $\int_{-1}^{0}(1-y^2)\cos(y-y^3/3)\ \mathrm{d}y$. Hooray, we now have the ingredient we need to get nice antiderivative for this function. This gives us $\sin(y-y^3/3)\|_{-}1^0 = \sin(2/3)$.

5. In this problem you'll compute the triple integral $\int\int\int_{\mathcal{W}} y\ \mathrm{d}V$ where $\mathcal{W}$ is the volume bounded between the paraboloids $z = x^2 + y^2$ and $z = 4 - (x^2 + y^2)$.

   (a) Draw a sketch of this region.



Solution:

(b) Observe that this region is z-simple and compute the projection of the region in the xy-plane.

Solution: We need to find where these paraboloids intersect and project this to the xy-plane. To do this, we rearrange $x^2 + y^2 = 4 - (x^2 + y^2)$ to get that $x^2 + y^2 = (\sqrt{2})^2$, so the projection to the xy-plane is the circle of radius $\sqrt{2}$.
Let's go ahead and describe this region as z-simple: it's the region where $x^2 + y^2 \leq z \leq 4 - (x^2 + y^2)$ and $-\sqrt{2 - x^2} \leq y \leq \sqrt{2 - x^2}$ and $-\sqrt{2} \leq x \leq \sqrt{2}$.

(c) Write the integral $\int\int\int_{\mathcal{W}} y \, \mathrm{d}V$ as an iterated integral. Since you described the region as z-simple you will write this integral as $\mathrm{d}z \, \mathrm{d}x \, \mathrm{d}y$ or $\mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x$.

Solution: Using our description of the region as being z-simple the integral is

$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-(x^2+y^2)} y \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}z.$$

(d) Compute the integral $\int\int\int_{\mathcal{W}} y \, \mathrm{d}V$.

Solution: After we compute the inner integral we are left with $\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} y(4 - 2x^2 - 2y^2) \, \mathrm{d}y \, \mathrm{d}x$.

We see that since our function is odd in y and even in x, if we integrate with respect to y first then we'll have an even degree function in y and our square root signs will all get taken care of. So, doing it in the order $\mathrm{d}y \, \mathrm{d}x$ is easier.

However, here is a way to compute it that is easier still: note that our region is symmetric out the xz-plane, i.e. if $(x, y, z) \in \mathcal{W}$ then so is $(x, -y, z)$. But we are just integrating the function $f(x, y, z) = y$, and obviously $-f(x, y, z) = f(x, -y, z)$, so everything cancels and we're left with zero.