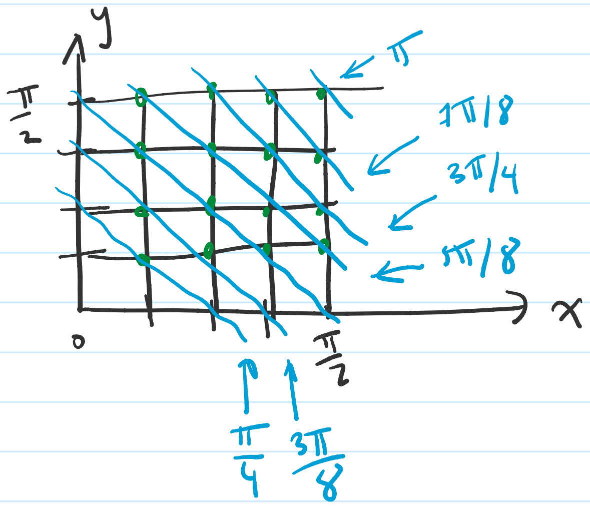
Worksheet 1 Solutions

2. Consider the surface defined by $z = \sin(x+y)$ over the rectangle $\mathcal{R} = [0, \pi/2] \times [0, \pi/2]$. Use a double Riemann sum with $m = n = 4$ to approximate the volume under the surface using lower left corners as sample points and upper right quarters as sample points.



Green – upper right sample points

Blue – where $\sin(x+y)$ is constant.

Summing $f(x,y)$ at sample points is given by:

$$S = \sin\left(\frac{\pi}{4}\right) + 2\sin\left(\frac{3\pi}{8}\right) + 3\sin\left(\frac{\pi}{2}\right) + 4\sin\left(5\pi/8\right) + 3\sin\left(3\pi/4\right) + 2\sin\left(7\pi/8\right) + \sin(\pi)$$

$$= \frac{\sqrt{2}}{2} + 4\sin\left(\frac{3\pi}{8}\right) + 7\sin\left(5\pi/8\right) + 3$$

since $\sin(x) = \sin(\pi - x)$.

$$\simeq 4.029$$

Each subdivision has area $(\pi/8)^2 = \pi^2/64$.

Hence the Riemann sum is given by

$$R = \frac{\pi^2}{64}\left(\frac{\sqrt{2}}{2} + 4\sin\left(\frac{3\pi}{8}\right) + 7\left(5\pi/8\right) + 3\right)$$

$$\simeq 0.621$$

The process is similar for the lower left hand corner.

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3. Let $\mathcal{R} = [a, b] \times [c, d]$ be a rectangle in the plane. Find $\int\int_{\mathcal{R}} 1 dA$ and $\int\int_{\mathcal{R}} k dA$ where $k \in \mathbb{R}$ is a constant.

The double integral $\iint_R f(x,y)dA$ is the signed volume under the function $f(x,y)$ over the rectangle R.

so $\iint_R 1dA = (b-a)(d-c)$

$\iint_R k dA = k(b-a)(d-c)$

4. Based on your answer to the previous question, if $\mathcal{R}$ is any shape in the plane what is $\int\int_{\mathcal{R}} 1 dA$?

It is the area of R.

5. Consider the rectangle $\mathcal{R} = [0, 1] \times [-1, 1]$. For which of the following functions is $\int\int_{\mathcal{R}} f(x, y) dA = 0$?

(1) $f(x, y) = e^{x^2+y^2} x$
(2) $f(x, y) = \cos(x + y) \sin(xy)$
(3) $f(x, y) = \cos(xy) \sin(xy)$
(4) $f(x, y) = xy^2$

In general: If $f(x,y)$ is odd in the x-variable (that is $f(-x,y) = -f(x,y)$) and R is a domain symmetric along the x-axis. (That is, reflective around the y-axis). Then $\iint_R f dA = 0$. This also works if we interchange x, y in the above statement.

(1) This is odd in the x-variable but R isn't symmetric along the x-axis.

In fact since $f(x,y) > 0$ for $x > 0$, we get that $\iint_R f\,dA > 0$.

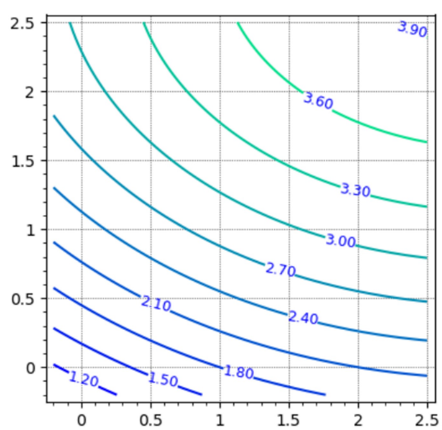
(2) $f(x,y)$ is not odd in either variable

(3) This is odd in the y-variable and R is symmetric along the y-axis. So $\iint_R f\,dA = 0$.

(4) This is odd in x but not R not symmetric along the x-axis.

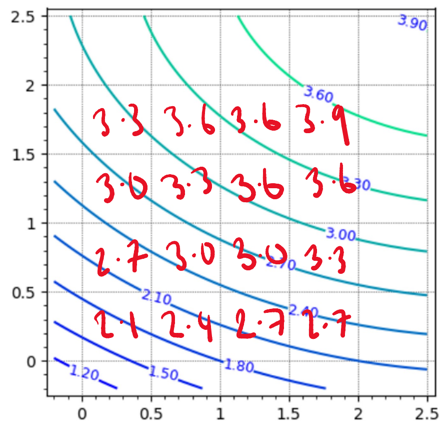
Hence we conclude the correct answer is (3).

6. Consider the following contour plot of a function $h(x,y)$.



Use Riemann sums with $m = n = 4$ to estimate $\iint_{[0,2]\times[0,2]} h(x,y)\,dA$. Give an upper and a lower bound for $\iint_{[0,2]\times[0,2]} h(x,y)\,dA$.

To find an upper bound, we need to find a value larger than $h(x,y)$ in each subdivision. We will make this be the height of the rectangular prism over that subdivision and so this rectangular prism will have more volume than the volume under $h(x,y)$ over that subdivision. Summing these up then gives an will have more volume than the volume under $h(x,y)$ over that subdivision. Summing these up then gives an upper bound.



For each square, we pick the value to be the height of the nearest countour line outside the square that is larger.

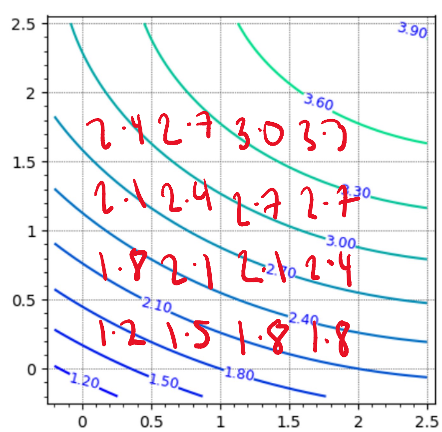
Each square has area $1/4$.

we then get our upper bound as:

$$U = \frac{1}{4}\left(2.1 + 2.4 + 3\times 2.7 + 3\times 3.0 + 3\times 3.3 + 4\times 3.6 + 3.9\right)$$

$$= 12.45$$

We do the same but pick the smaller points for the lower bound.



$$L = \frac{1}{4}\left(1.2 + 1.5 + 3\times 1.8 + 3\times 2.1 + 3\times 2.4 + 3\times 2.7 + 3.0 + 3.3\right)$$

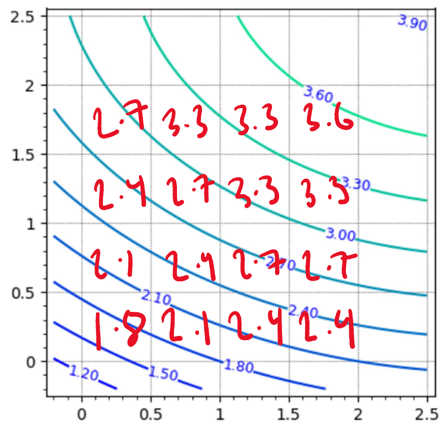
$$= 9.$$

Hence we have that

Hence we have that

$$9 \leq \iint_{[0,2]\times[0,2]} h(x,y)\,dA \leq 12.45.$$

We can also construct a Riemann sum to approximate it. For this we pick any value inside each subdivision to use as sample points. ie, we can pick the value of a contour line as our sample values.



$$R = \frac{1}{4}\left(1.8 + 2\times 2.1 + 4\times 2.4 + 4\times 2.7 + 4\times 3.3 + 3.6\right)$$

$$= 10.8.$$

So $$\iint_{[0,2]\times[0,2]} h(x,y)\,dA \approx 10.8.$$