Remember Stoke's theorem: if $\mathcal{S}$ is an oriented surface and $\partial\mathcal{S}$ has the boundary orientation then if $\mathbf{F}$ is a vector field with continuous partial derivative then $\iint_{\mathcal{S}} \nabla \times \mathbf{F} \cdot dS = \oint_{\partial\mathcal{S}} \mathbf{F} \cdot dr$.

1. Let $\mathbf{F}$ be the vector field $\langle x, y, xyz\rangle$ and let $\mathcal{S}$ be the part of the plane $2x + y + z = 2$ that lies in the first octant oriented upwards. Verify that Stoke's theorem holds in this example by explicity computing $\iint_{\mathcal{S}} \nabla \times F \cdot dS$ and $\oint_{\partial\mathcal{S}} F \cdot dr$.

2. Let $\mathcal{S}_1$ be the surface $x^2+y^2+4z^2 = 4$ where $z \geq 0$ and let $\mathcal{S}_2$ be the surface $z = 4 - \overset{x^2}{\cancel{y^2}} - y^2$ where $z \geq 0$, where each surface is oriented with the normal pointed upwards. If $\mathbf{F}$ is a vector field with continuous partial derivatives explain why $\iint_{\mathcal{S}_1} \nabla \times F \cdot dS = \iint_{\mathcal{S}_2} \nabla \times F \cdot dS$.

3. (a) Let $\mathcal{D}$ be the disc $x^2 + y^2 \leq 4$ with upward pointing orientation and let $\mathbf{F}$ be the vector field $\mathbf{F} = \langle xz\sin(yz), \cos(yz), e^{x^2+y^2}\rangle$. What is $\iint_{\mathcal{D}} \mathbf{F} \cdot dS$?

   (b) Let S be the part of the paraboloid $z = 4 - x^2 - y^2$ with $z \geq 0$ with downward pointing orientation. What is $\iint_S \mathbf{F} \cdot dS$ (here $\mathbf{F}$ is the vector field from the previous part of the problem)? **Hint:** Does $\mathbf{F}$ have a vector potential?.

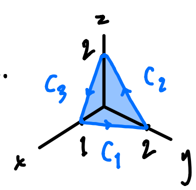
4. Let $\mathcal{W}$ be the part of the solid cylinder $x^2 + y^2 \leq 1$ where $0 \leq z \leq 1$, let $\partial\mathcal{W}$ be the boundary of this solid with the outwards pointing orientation, and let $\mathbf{F} = \langle xy, yz, xz\rangle$.

   (a) Directly compute $\iint_{\partial\mathcal{W}} \mathbf{F} \cdot dS$.

   (b) Directly compute $\iiint_{\mathcal{W}} \operatorname{div} \mathbf{F}\, dV$.

   (c) Compare your answers– what do you notice?

1.



$z = 2 - 2x - y \qquad S(x, y) = (x, y, 2 - 2x - y)$

$\vec{F} = (x, y, xyz)$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x & y & xyz \end{vmatrix} = (xz, -yz, 0)$$

$-C_1$: $(t, 2-2t, 0)$ $[0,1]$

$-C_2$: $(0, t, 2-t)$ $[0,2]$

C_3: $(t, 0, 2-2t)$ $[0,1]$

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_0^1\int_0^{2-2x} (x(2-2x-y), -y(2-2x-y), 0) \cdot \vec{N}\, dy\, dx$$

where $\vec{\partial_x S} = (1, 0, -2)$, $\vec{\partial_y S} = (0, 1, -1)$

$\Rightarrow \vec{N} = \vec{\partial_x S} \times \vec{\partial_y S} = (2, 1, 1)$

$$= \int_0^1\int_0^{2-2x} 4x - 4x^2 - 2y + y^2\, dy\, dx = 0$$

On the other hand,

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = -\int_0^1 (t, 2-2t, 0) \cdot (1, -2, 0)\, dt - \int_0^2 (0, t, 0) \cdot (0, 1, -1)\, dt$$

$$+ \int_0^1 (t,0,0)\cdot(1,0,-2)\,dt$$

$$= \int_0^1 \cancel{-t} + 2(2-2t)\,dt - \int_0^2 t\,dt + \cancel{\int_0^1 t\,dt}$$

$$= 4 - 4\underbrace{\int_0^1 t\,dt}_{\frac{1}{2}} - \underbrace{\int_0^2 t\,dt}_{2} = 0, \text{ as desired.}$$

2. We see by taking $z=0$ that both S_1, S_2 are bounded by the circle $x^2+y^2=4$, both with positively oriented (i.e. CCW) boundary.

Thus by Stokes' thm.,

$$\iint_{S_1} (\nabla \times \vec{F})\cdot d\vec{S} = \int_{\partial S_1} \vec{F}\cdot d\vec{r} = \int_{\partial S_2} \vec{F}\cdot d\vec{r} = \iint_{S_2} (\nabla\times\vec{F})\cdot d\vec{S}.$$

3a. $\iint_D \vec{F}\cdot d\vec{S}$

$D(x,y) = (x,y,0)$

$\partial_x D = (1,0,0), \quad \partial_y D = (0,1,0)$

$\vec{N} = (0,0,1)$

$$= \iint_D (0,1,e^{x^2+y^2})\cdot(0,0,1)\,dA \overset{\text{polar}}{=} \int_0^{2\pi}\int_0^2 re^{r^2}\,dr\,d\theta$$

$$= 2\pi\left(\tfrac{1}{2}e^{r^2}\right)\Big|_0^2 = \cancel{2}\pi\cdot\tfrac{1}{\cancel{2}}(e^4-1) = \pi(e^4-1).$$

b. Recall that Stokes' thm applies to the integral of the curl of a vec. field. So we want to write $\vec{F}$ as the curl of another field, $\vec{F} = \nabla\times\vec{A}$ for some $\vec{A}$.

Note that $\nabla\cdot\vec{F} = \partial_x F_1 + \partial_y F_2 + \partial_z F_3$

$$= z\sin yz - z\sin yz + 0 = 0$$

so since $\vec{F}$ is defined on $\mathbb{R}^3$, $\vec{F}$ has a vector potential $\vec{A}$ (that is, $\vec{F} = \nabla\times\vec{A}$)

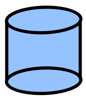
(see also: Poincaré's lemma)

Hence by Stokes' thm.,

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S (\nabla\times\vec{A})\cdot d\vec{S} = \int_{\partial S} \vec{A}\cdot d\vec{r} = -\int_{\partial D} \vec{A}\cdot d\vec{r}$$

$$= -\iint_D \vec{F} \cdot d\vec{S} = -\pi(e^4 - 1) \quad \text{from part (a).}$$

4a. For the side, $S(u, v) = (\cos u, \sin u, v) \qquad 0 \le u \le 2\pi,\ 0 \le v \le 1$



$$\partial_u S = (-\sin u, \cos u, 0)$$

$$\partial_v S = (0, 0, 1)$$

$$\vec{N} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos u, \sin u, 0)$$

$$\iint_{\text{side}} \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^1 (\cos u \sin u, v \sin u, v \cos u) \cdot (\cos u, \sin u, 0)\, dv\, du$$

$$= \int_0^{2\pi}\int_0^1 \cos^2 u \sin u + v \sin^2 u \, dv\, du$$

$$= \int_0^{2\pi} \underbrace{\cos^2 u \sin u}_{\frac{d}{du}\left(-\frac{1}{3}\cos^3 u\right)}\, du + \frac{1}{2}\int_0^{2\pi} \sin^2 u\, du = \frac{\pi}{2}$$

For the top disc, $(r\cos\theta, r\sin\theta, 1)$ has $\vec{N} = (0, 0, r)$,

$$\iint_{\text{top}} \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^1 r^2 \cos\theta \, dr\, d\theta = 0$$

& sim. $\iint_{\text{bottom}} \vec{F} \cdot d\vec{S} = 0$. So $\iint_{\partial W} \vec{F} \cdot d\vec{S} = \frac{\pi}{2}$.

b. $\nabla \cdot \vec{F} = \partial_x(xy) + \partial_y(yz) + \partial_z(xz) = y + z + x$

$$\iiint_W (\nabla \cdot \vec{F})\, dV = \int_0^{2\pi}\int_0^1\int_0^1 (z + r(\cos\theta + \sin\theta))\, r\, dz\, dr\, d\theta = \frac{\pi}{2}.$$

c. These are the same (which also follows from the Divergence Theorem).