Midterm 2 practice

UCLA: Math 32B, Fall 2019

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Version: practice

- This exam has 4 questions, for a total of 26 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

Question	Points	Score
1	10	
2	8	
3	8	
4	0	
Total:	26	

1. (a) (5 points) Let $\mathcal{D}$ be the region in the xy-plane above the x-axis and below the curve $y = 1 - x^2$. Compute the integrals

$$I_1 = \frac{1}{A}\iint_{\mathcal{D}} x\ dA \text{ and } I_2 = \frac{1}{A}\iint_{\mathcal{D}} y\ dA$$

where A is the area of $\mathcal{D}$.

Solution: We describe $\mathcal{D}$ as a vertically simple region

$$\mathcal{D} = \{\ (x, y)\ \mid\ -1 \leq x \leq 1, 0 \leq y \leq 1 - x^2\ \}$$

We first compute the area

$$A = \iint_{\mathcal{D}} 1\ dA = \int_{-1}^{1}\int_{0}^{1-x^2} 1\ dy\ dx = \int_{-1}^{1} 1 - x^2\ dx = 2 - 2/3 = 4/3.$$

Now we compute

$$\iint_{\mathcal{D}} x\ dA = \int_{-1}^{1}\int_{0}^{1-x^2} x\ dy\ dx = \int_{-1}^{1}(1 - x^2)x\ dx = 0.$$

Now we compute

$$\iint_{\mathcal{D}} y\ dA = \int_{-1}^{1}\int_{0}^{1-x^2} y\ dy\ dx = \int_{-1}^{1}\frac{1}{2}(1 - x^2)^2\ dx = 16/30.$$

So

$$I_1 = 0 \text{ and } I_2 = 2/5.$$

(b) (5 points) Find a parameterisation $\mathbf{r}(t)$, of the curve that is the intersection of the surfaces $y = x^2$ and $x + y + z = 1$, oriented from $x = -4$ to $x = 4$, such that $t \in [0, 1]$ What is the velocity of the parameterisation?

Solution: Suppose we set $x = t$, then it is clear that $y = t^2$ and so $t + t^2 + z = 1$ so we get the parameterisation

$$\mathbf{s}(t) = (t, t^2, 1 - z - z^2)$$

but in this parameterisation we have $t \in [-4, 4]$. We can adjust this by considering $8t - 4$. As t ranges from 0 to 4, the value $8t - 4$ would range from -4 to 4. So the parameterisation we use is

$$\mathbf{r}(t) = \mathbf{s}(8t - 4) = (8t - 4, (8t - 4)^2, 1 - (8t - 4) - (8t - 4)^2)$$

The velocity is

$$\mathbf{r}'(t) = (8, 16(8t - 4), 8 - 16(8t - 4))$$

2. (8 points) Consider the region $\mathcal{E}$ given by

$$0 \le z \le (y - x^2)^2, \quad x^2 \le y \le x.$$

Use the change of variables

$$x = u, y = v + u^2, z = wv^2,$$

to evaluate

$$\iiint_{\mathcal{E}} \frac{1}{y - x^2} \, \mathrm{d}V.$$

Solution: First we describe $\mathcal{E}$ in the form

$$\mathcal{E} = \{ \ (x, y, z) \ \mid \ (x, y) \in \mathcal{D} \text{ and } 0 \le z \le (y - x^2)^2 \ \}$$

where

$$\mathcal{D} = \{ \ (x, y) \ \mid \ 0 \le x \le 1 \text{ and } x^2 \le y \le x \ \}.$$

Our next job is to figure out which region in uvw-space is mapped to $\mathcal{E}$ when we apply $G(u, v, w) = (u, v + u^2, wv^2)$. We can use the inequalities given, in terms of u, v, w.

$$0 \le wv^2 \le v^2, u^2 \le v + u^2 \le u.$$

We can manipulate these to

$$0 \le w \le 1, \text{ and } 0 \le v \le u - u^2.$$

Thus if we take

$$\mathcal{E}' = \{ \ (u, v, w) \ \mid \ (u, v) \in \mathcal{D}' \text{ and } 0 \le w \le 1 \ \}$$

where

$$\mathcal{D}' = \{ \ (u, v) \ \mid \ 0 \le u \le 1, 0 \le v \le u - u^2 \ \}.$$

Now we need to find the Jacobian:

$$J(G) = \det \begin{pmatrix} 1 & 0 & 0 \\ 2u & 1 & 0 \\ 0 & 2vw & v^2 \end{pmatrix} = v^2$$

This is always positive! Thus

$$\begin{aligned}
\iiint_{\mathcal{E}} \frac{1}{y - x^2} \, dV &= \iiint_{\mathcal{E}'} \frac{1}{v} \|J(G)\| \, dV_{uvw} \\
&= \iiint_{\mathcal{E}'} v \, dV_{uvw} \\
&= \iint_{\mathcal{D}'} \int_0^1 v \, dw \, dA_{uv} \\
&= \int_0^1 \int_0^{u-u^2} \int_0^1 v \, dw \, dv \, du \\
&= \int_0^1 \int_0^{u-u^2} v \, dv \, du \\
&= \int_0^1 \frac{1}{2}(u - u^2)^2 \, du = 1/60
\end{aligned}$$

3. Let $\mathbf{F}$ be the vector field on $\mathbb{R}^3$ given by

$$\mathbf{F}(x, y, z) = (y\cos z - yze^x, x\cos z - ze^x, -xy\sin z - ye^x).$$

(a) (4 points) Show that $\mathbf{F}$ is conservative.

Solution: Our vector field is defined on a simply connected domain. This means being conservative is equivalent to having curl zero. So we simply check this:

$$\begin{aligned}\nabla \times \mathbf{F} &= \langle \partial_x, \partial_y, \partial_z \rangle \times \langle y\cos z - yze^x, x\cos z - ze^x, -xy\sin z - ye^x \rangle \\ &= \langle -x\sin z - e^x - (-x\sin z - e^x), -y\sin z - ye^x - (-y\sin z - ye^x), \cos z - ze^x - (\cos z - ze^x) \rangle = 0.\end{aligned}$$

(b) (4 points) Find a potential function for $\mathbf{F}$.

Solution: We need a function f such that

$$\begin{aligned}\partial_x f &= y\cos z - yze^x \\ \partial_y f &= x\cos z - ze^x \\ \partial_z f &= -xy\sin z - ye^x\end{aligned}$$

This means we get three conditions

$$\begin{aligned}f &= xy\cos z - yze^x + \alpha(y, z) \\ f &= xy\cos z - yze^x + \beta(x, z) \\ f &= xy\cos z = yze^x + \gamma(x, y)\end{aligned}$$

We can simply let $\alpha = \beta = \gamma = 0$ and take

$$f = xy\cos z - yze^x.$$

4. Consider the vector field $\mathbf{F} = \left\langle yze^{(xyz)^2}, xze^{(xyz)^2}, xye^{(xyz)^2} + 3z^2 \right\rangle$. Let $\mathbf{C}$ be the curve given by the intersection of the cylinder $x^2 + (y-1)^2 = 1$ and the surface $y = 1 - z^2$ and $x \geq 0$, oriented upwards. Calculate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$. You may use the fact that $\int_{-1}^{1} e^{t^2}\ dt =$ *Hint: You wont be able to evaluate the integral directly. You need another method.*

Solution: The field is defined everywhere in $\mathbb{R}^3$ which is simply connected, and has zero curl, so it is conservative. We could try and use the fact that if we find a potential function f then

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = f(Q) = f(P)$$

where P and Q are the start and end points. But the problem is that we will have an extremely difficult time trying to find a potential function. It seems impossible! So we try something easier. We know that since the field is conservative, we can integrate along any path that starts and ends at the same points. It isn't hard to see that the curve starts at $P = (0, 0, -1)$ and ends at $Q = (0, 0, 1)$. Let L be the straight line along the z-axis between these points oriented upwards. Then

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{L} \mathbf{F} \cdot d\mathbf{r}$$

We parameterise L by $\mathbf{r}(t) = (0, 0, t)$, with $t \in [-1, 1]$. Then $\mathbf{r}'(t) = (0, 0, 1)$ and $\mathbf{F}(\mathbf{r}(t)) = \langle 0, 0, 3t^2 \rangle$. Thus

$$\int_{L} \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^{1} 3t^2\ dt = \left[t^3\right]_{-1}^{1} = 2$$

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