Upload your solutions to gradescope for the following questions by 11:59pm LA time on Sunday 17 April.

- Late exams will not be accepted.
- Your scans must be readable and good quality. Use good lighting and a scanning app.
- Questions 1,2,3,4 must begin on a new page and questions must be allocated correctly on Gradescope.
- Write your solutions **linearly**. We should be able to easily read your solutions and do not want to hunt around the page for it.

1. The *twisted cubic* is the curve in $\mathbb{R}^3$ that is the intersection of the surfaces $y = x^2$ and $z = x^3$. Let $\mathcal{C}$ be the part of the twisted cubic where $x \in [0, 1]$. Let $f(x, y, z) = (1 + 4y + 9xz)^{-\frac{1}{2}}$.

   (a) (2 points) Find a parametrisation of the curve $\mathcal{C}$, making sure to indicate the range of t. *Hint: If (x, y, z) is a point on the curve where $x = t$, then $y = t^2$ and $z = \ldots$.*

   Solution: $\mathbf{r}(t) = (t, t^2, t^3)$ for $t \in [0, 1]$.

   (b) (3 points) Evaluate the integral

   $$\int_{\mathcal{C}} f(x, y, z)\ ds$$

   Solution: We know that $\int_{\mathcal{C}} f(x, y, z)\ ds = \int_0^1 f(r(t))||r'(t)||\ dt$. Since $r'(t) = \langle 1, 2t, 3t^2\rangle$ we see that $||r'(t)|| = \sqrt{1 + 4t^2 + 9t^4}$. We also see that $f(r(t)) = \frac{1}{\sqrt{1+4t^2+9t^4}}$. So, $\int_{\mathcal{C}} f(x, y, z)\ ds = \int_0^1 1\ dt = 1$.

2. Consider the solid ellipsoid E given by $(x/2)^2 + (y/3)^2 + (z/4)^2 \leq 1$ measured in meters with density function given by $\delta(x, y, z) = \sqrt{(x/2)^2 + (y/3)^2 + (z/4)^2}\ kg/m^3$.

   (a) (2 points) Find a change of coordinates G so that G applied to the solid ball of radius 1 centered at the origin gives E. *Hint: your map G should change the equation for a sphere into the equation of the ellipsoid.*

   Solution: The coordinate change with $u = x/2$, $v = y/3$, $w = z/4$ does the trick. So, we have that $G(u, v, w) = (2u, 3v, 4w)$. Substituting u, v, w into our equation for the ellipse gives the solid ball as desired.

   (b) (4 points) What is the mass of E?

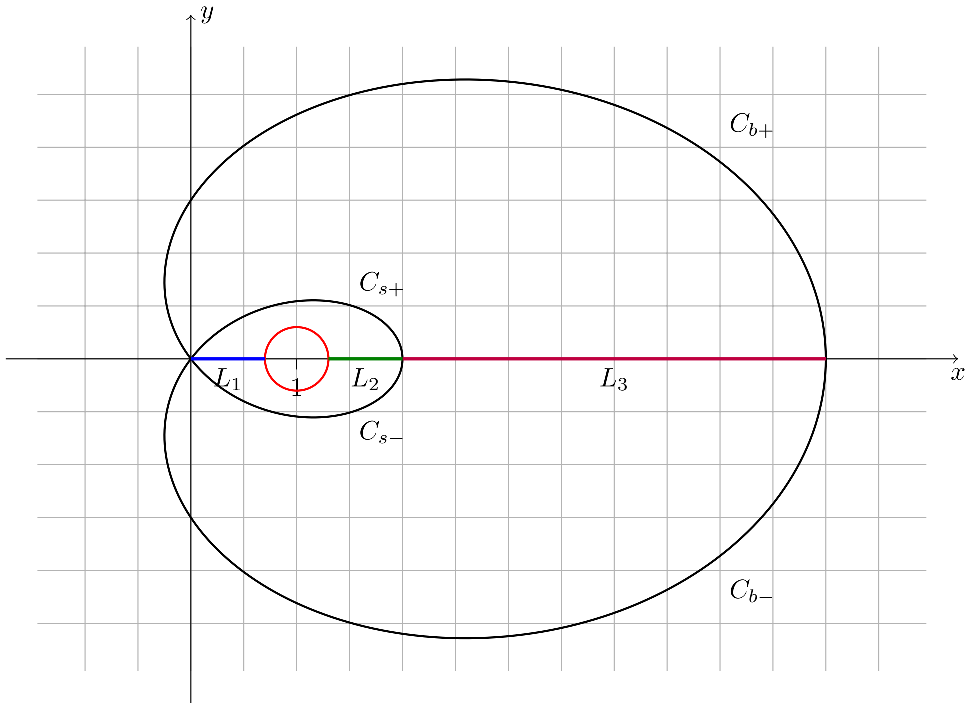
   Solution: We want to compute $\iiint_E \sqrt{(x/2)^2 + (y/3)^2 + (z/4)^2}\ dV$. By the change of coordinates theorem this is $\iiint_B \sqrt{u^2 + v^2 + w^2}|J(G)|\ dV$ where B is the solid ball of radius one centered at the origin. The Jacobian of G is 24, so after converting the second integral to spherical coordinates we have that

   $$\iiint_E \sqrt{(x/2)^2 + (y/3)^2 + (z/4)^2}\ dV = 24 \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^3 \sin\phi\ d\rho\ d\phi\ d\theta.$$

   This is readily computed to give that the mass is 24π kilograms.

3. In this question assume that $\mathbf{E}$ is a vector field defined on the whole of $\mathbb{R}^2$, apart from the point $(1, 0)$. Suppose that $\nabla \times \mathbf{E} = 0$. The function $\mathbf{r}(t) = \big(2\cos t + 4\cos^2 t, 2\sin t + 4\cos t \sin t\big)$ for $t \in [-\frac{2\pi}{3}, \frac{4\pi}{3}]$

traces out the curve $\mathcal{C}$ on the graph below. To give you an idea: it starts at the origin, traces out the large loop, returns to the origin when $t = 2\pi/3$, then traces out the small loop and then returns to the origin once more.



(a) (2 points) Redraw the above graph and indicate the orientation of the curve. *Hint: calculating some tangent vectors might help.*

> **Solution:** The curve is oriented counterclockwise about both loops.

(b) (4 points) Let $\mathcal{A}$ be the circle radius $\frac{1}{2}$, and centre $(1, 0)$ (so it fits entirely within the small loop above) oriented counter clockwise. Suppose that

$$\int_{\mathcal{A}} \mathbf{E} \cdot d\mathbf{r} = 2$$

What is $\int_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{r}$? Justify your answer carefully, the answer itself will only be worth 1 point.

> **Solution:** The strategy will be to use the fact that the curl of $\mathbf{E}$ is zero. So on any simply connected domain $\mathbf{E}$ is conservative and thus integrals of $\mathbf{E}$ are path independent in these domains. The two domains we will use are the upper and lower half planes. Accordingly we will split up the curve $\mathcal{C}$ into four parts, first the big loop C_b into C_{b+} and C_{b-} the parts above and below the x-axis respectively. Similarly the small loop, C_s is split into C_{s+} and C_{s-}. We also split $\mathcal{A}$ into A_+ and A_-. Thus we have $\mathcal{C} = C_{b-} + C_{b+} + C_{s-} + C_{s+}$. We will come up with an expression for the integral of each of the four parts.
>
> First we concentrate on C_{b+}. To relate this to A, let L_1, L_2 and L_3 be the lines indicated above in the diagram, with orientations pointing to the left. Thus by path independence
>
> $$\int_{C_{b+}} \mathbf{E} \cdot d\mathbf{r} = \int_{L_3+L_2+A_++L_1} \mathbf{E} \cdot d\mathbf{r} = \int_{L_3} \mathbf{E} \cdot d\mathbf{r} + \int_{L_2} \mathbf{E} \cdot d\mathbf{r} + \int_{A_+} \mathbf{E} \cdot d\mathbf{r} + \int_{L_1} \mathbf{E} \cdot d\mathbf{r}$$

We get similar expressions

$$\int_{C_{b-}} \mathbf{E}\cdot d\mathbf{r} = \int_{-L_1+A_--L_2-L_3} \mathbf{E}\cdot d\mathbf{r} = -\int_{L_1} \mathbf{E}\cdot d\mathbf{r} + \int_{A_-} \mathbf{E}\cdot d\mathbf{r} - \int_{L_2} \mathbf{E}\cdot d\mathbf{r} - \int_{L_3} \mathbf{E}\cdot d\mathbf{r}$$

$$\int_{C_{s+}} \mathbf{E}\cdot d\mathbf{r} = \int_{L_2+A_++L_1} \mathbf{E}\cdot d\mathbf{r} = \int_{L_2} \mathbf{E}\cdot d\mathbf{r} + \int_{A_+} \mathbf{E}\cdot d\mathbf{r} + \int_{L_1} \mathbf{E}\cdot d\mathbf{r}$$

$$\int_{C_{s-}} \mathbf{E}\cdot d\mathbf{r} = \int_{-L_1+A_--L_2} \mathbf{E}\cdot d\mathbf{r} = -\int_{L_1} \mathbf{E}\cdot d\mathbf{r} + \int_{A_-} \mathbf{E}\cdot d\mathbf{r} - \int_{L_3} \mathbf{E}\cdot d\mathbf{r}$$

Adding these all up gives

$$\oint_{\mathcal{C}} \mathbf{E}\cdot d\mathbf{r} = \int_{A} \mathbf{E}\cdot d\mathbf{r} + \int_{A} \mathbf{E}\cdot d\mathbf{r} = 2+2 = 4$$

4. (6 points) Consider the vortex field $\mathbf{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ and the curve $\mathcal{C}$ given by $y = x^4 - 14$ where $-2 \le x \le 2$ and the curve is oriented from left to right. Evaluate

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}.$$

Solution: If we try and do this directly, we would use the parameterisation (t, t^4-14) for $t \in [-2,2]$ and get the integral

$$\int_{-2}^{2} \frac{14+3t^4}{t^2+(t^4-14)^2}\,dt = \int_{-2}^{2} \frac{14+3t^4}{196+t^2-28t^4+t^8}\,dt$$

which will be very difficult to do even by partial fractions. We need to find another way.

We will introduce a new curve $\mathcal{H}$ which is the circle $x^2+y^2=8$ restricted to those points where $\theta \in [\pi/4, 3\pi/4]$, oriented right to left. Then $\mathcal{C}+\mathcal{H}$ is a loop that goes around the origin. Standard arguments mean that

$$\int_{\mathcal{C}+\mathcal{H}} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} + \int_{\mathcal{H}} \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} = 2\pi$$

where C_1 is the counter clockwise circle about the origin of radius one. Thus

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = 2\pi - \int_{\mathcal{H}} \mathbf{F}\cdot d\mathbf{r}.$$

We can parameterise $\mathcal{H}$ by $\mathbf{r}(t) = (2\sqrt{2}\cos t, 2\sqrt{2}\sin t)$ for $t \in [\pi/4, 3\pi/4]$. Then $\mathbf{r}'(t) = \langle -2\sqrt{2}\sin t, 2\sqrt{2}\cos t\rangle$, and noting that $r^2 = 8$ we get

$$\int_{\mathcal{H}} \mathbf{F}\cdot d\mathbf{r} = \frac{1}{8}\int_{\pi/4}^{3\pi/4} 8\sin^2 t + 8\cos^2 t\,dt = \frac{\pi}{2}$$

Hence

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \frac{3\pi}{2}.$$