UCLA: Math 32B   Midterm 1 - part 2 (solutions)   Spring, 2020

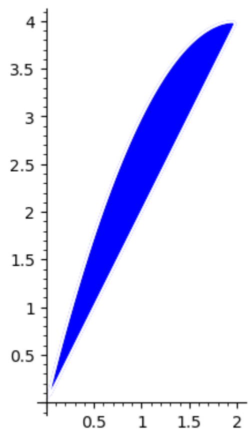
Upload your solutions to gradescope for the following questions by 11:59pm LA time on Wednesday 22 April.

- Late exams will not be accepted.
- Your scans must be readable and good quality. Use good lighting and a scanning app.
- Questions 1,2,3 must begin on a new page and questions must be allocated correctly on Gradescope.
- Write your solutions **linearly**. We should be able to easily read your solutions and do not want to hunt around the page for it.
- Be sure to show your work and justify your answers.

1. 1. In this question we will first consider the region $\mathcal{C}$ that is bounded by $y = x(4-x)$ and $y = 2x$.

   (a) (1) Sketch the region clearly.

   Solution:



   (b) (2) Write the integral

$$\iint_{\mathcal{C}} x \, dA$$

   as iterated integrals in both orders $dx\, dy$ and $dy\, dx$ (no need to evaluate the integral).

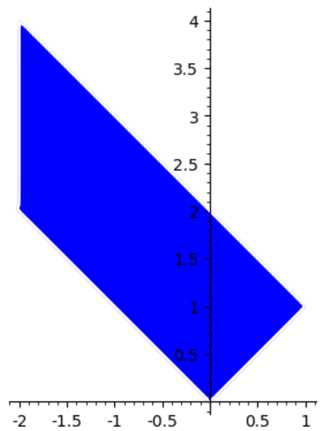
   Solution:
   Note that these curves intersect at $(0,0)$ and at $(2,4)$, and that when $0 \le x \le 2$ the curve $y = 2x$ is below the curve $y = x(4-x)$.
   When we solve these functions for x we get that $x = y/2$ and (since $y = x(4-x) = -(x-2)^2 + 4$) thant $x = \pm\sqrt{4-y} + 2$. In this region (since x is ranging from 0 to 2) we need to pick the negative branch of the square root.
   So, the integral is $\int_0^2 \int_{2x}^{x(4-x)} x \text{ dy dx}$ and $\int_0^4 \int_{-\sqrt{4-y}+2}^{y/2} x \text{ dx dy}$.

2. Now, consider the region $\mathcal{D}$ given by the inequalities

   - $0 \le x + y \le 2$,
   - $0 \le y - x$, and
   - $-2 \le x$.

   (a) (1) Sketch the region clearly in the xy-plane.

Solution:



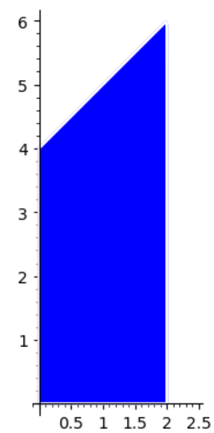
(b) (3) Let $u = x + y$ and $v = y - x$, and use this change of coordinates to express the integral

$$\iint_{\mathcal{D}} 2x + 2y \, dA$$

as an iterated integral of u and v (no need to evaluate the integral). Make sure you sketch the region in the uv-plane that you are integrating over. *Hint: the region you will be integrating over in the uv-plane is not a rectangle.*

Solution: Since $u = x + y$ and $v = y - x$, we see that $(u - v) = 2x$. So, in the uv-plane our region is $0 \leq u \leq 2$, $0 \leq v$, and $-2 \leq (u - v)/2$, so $v \leq u + 4$.

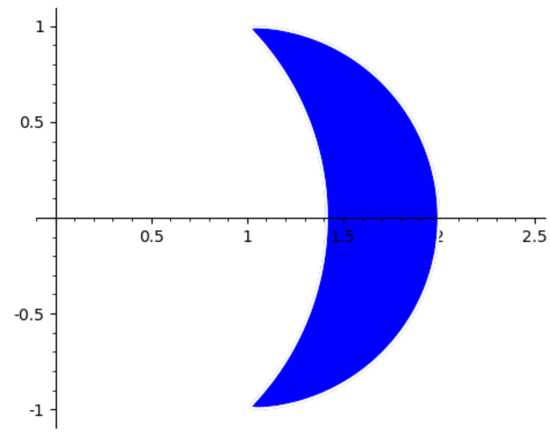
So, our region looks like:



The Jacobian of this change of the inverse change of coordinates is 2, so the Jacobian of this change of coordinates is 1/2 and our integral is $\int_0^2 \int_0^{u+4} (u) \, dv \, du$.

2. In this question we will consider the region $\mathcal{D}$ given by $-1 \leq y \leq 1$ and $\sqrt{2 - y^2} \leq x \leq 1 + \sqrt{1 - y^2}$.

   (a) (2) Sketch the region $\mathcal{D}$ in the xy-plane. Make sure your drawing is large and clear.



Solution:

(b) (3) Write the area of $\mathcal{D}$ as an interated integral in *polar coordinates* (no need to evaluate the integral).

Solution: Note that our region can be desribed as the points inisde the circle $(x-1)^2+y^2=1$ and outside the circle $x^2+y^2=2$. In polar coordinates these curves are respectively $r=2\cos\theta$ and $r=\sqrt{2}$. The θ values where these circles intersect are solutions to $2\cos(\theta)=\sqrt{2}$, so at $\theta=\pm\pi/4$. So, the area is $\int_{-\pi/4}^{\pi/4}\int_{\sqrt{2}}^{2\cos\theta} r \, \mathrm{d}r \, \mathrm{d}\theta$.

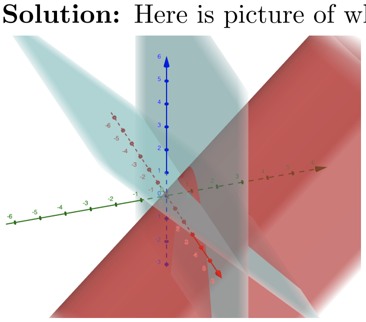
3. Consider the region $\mathcal{E}$ that is bounded by the surface $z=y-x^2$ and the planes $1-y-z=0$ and $y=1$.

(a) (2) Describe the region in the form

$$\mathcal{E}=\{\,(x,y,z)\in\mathbb{R}^3 \mid (x,y)\in\mathcal{D},\ z_1(x,y)\le z\le z_2(x,y)\}$$

for $\mathcal{D}$ a region in the xy-plane. Your answer should specify what $\mathcal{D}$ is.

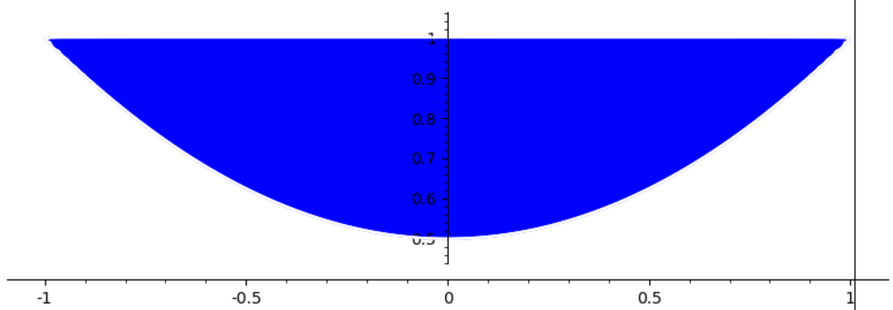
Solution: Here is picture of what these three surfaces look like.



We need to figure out where these surfaces intersect. The surfaces $z=y-x^2$ and $z=1-y$ intsersect where (x,y) satisfy $1-y=y-x^2$, or $y=\frac{1+x^2}{2}$. Note alo that the planes $1-y-z=0$ and $y=1$ intersect in the xy-plane along the line $y=1$.

So, the projection of this region to the xy plane is bounded by $y=\frac{1+x^2}{2}$ and $y=1$, which we can describe as the region $\mathcal{D}=\{(x,y): -1\le x\le 1, \frac{1+x^2}{2}\le y\le 1$. Note that over $\mathcal{D}$ that since $\frac{1+x^2}{2}\le y$, then $1+x^2\le 2y$, so $1-y\le y-x^2$, therefore over this region we conclude that $1-y\le z\le y-x^2$.

So, we can describe this region as $\{(x,y,z):(x,y)\in\mathcal{D}, 1-y\le z\le y-x^2$.



Here is a picture of $\mathcal{D}$.

(b) (3) Compute the volume of the region $\mathcal{E}$.

Solution: To compute the volume of $\mathcal{E}$ we just need to compute the integral $\iiint_{\mathcal{E}} 1\, dV$.

$$
\begin{aligned}
\text{Vol}(\mathcal{E}) &= \iiint_{\mathcal{E}} 1\, dV \\
&= \iint_{\mathcal{D}} \int_{1-y}^{y-x^2} 1\, dz\, dA \\
&= \int_{-1}^{1} \int_{\frac{1+x^2}{2}}^{1} \int_{1-y}^{y-x^2} 1\, dz\, dy\, dx \text{ (write as an iterated integral)} \\
&= \int_{-1}^{1} \int_{\frac{1+x^2}{2}}^{1} 2y - x^2 - 1\, dy\, dx \text{ (compute the first integral)} \\
&= \int_{-1}^{1} y^2 - x^2 y - y\Big|_{y=\frac{x^2+1}{2}}^{1}\, dx \text{ (compute the second integral)} \\
&= \int_{-1}^{1} -x^2 - \left(\frac{(x^2+1)^2}{4} - x^2\frac{1+x^2}{2} - \frac{1+x^2}{2} \right)\, dx \\
&= \int_{-1}^{1} x^4/4 - x^2/2 + 1/4\, dx \text{ (simplify)} \\
&= x^5/20 - x^3/6 + x/4\Big|_{-1}^{1} \\
&= 2/20 - 2/6 + 1/2 \\
&= 4/15
\end{aligned}
$$