Final exam (practice 1)

UCLA: Math 32B, Fall 2019

Instructor: Noah White
Date:

- This exam has 7 questions, for a total of 80 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

Question	Points	Score
1	12	
2	12	
3	12	
4	13	
5	10	
6	11	
7	10	
Total:	80	

Questions 1 and 2 are multiple choice. Once you are satisfied with your solutions, indicate your answers by marking the corresponding box in the table below.

Please note! The following four pages will not be graded. You must indicate your answers here for them to be graded!

Question 1.

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				
(f)				

Question 2.

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				
(f)				

Here are some formulas that you may find useful as some point in the exam.

$$\int \cos^2 x \ \mathrm{d}x = \frac{1}{2}\left(x + \cos x \sin x\right)$$

$$\int \sin^2 x \ \mathrm{d}x = \frac{1}{2}\left(x - \cos x \sin x\right)$$

$$\int \sin x \cos x \ \mathrm{d}x = \frac{1}{2}\sin^2 x$$

$$\int_0^\pi \sin^3 \ dx = \frac{4}{3}$$

Spherical coordinates are given by

$$\begin{aligned} x(\rho,\theta,\phi) &= \rho\cos\theta\sin\phi \\ y(\rho,\theta,\phi) &= \rho\sin\theta\sin\phi \\ z(\rho,\theta,\phi) &= \rho\cos\phi \end{aligned}$$

The Jacobian for the change of coordinates is $J = \rho^2 \sin\phi$.

The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

   (a) (2 points) Calculate the curl of $\langle -y, x, y - x\rangle$

   A. $\langle 1, 1, 2\rangle$
   B. $\langle 1, 2, 1\rangle$
   C. $\langle 2, 1, 1\rangle$
   D. $\langle 0, 0, 1\rangle$

   (b) (2 points) Integrate $f(x, y) = x^3 \sin y$ in the region $\mathcal{R} = [-1, 1] \times [0, 4]$

   A. π
   B. $-\pi$
   C. 1
   D. 0

   (c) (2 points) Integrate the function $f(x, y) = 12xy$ on the region $\mathcal{D}$ bounded by $y = x^2$ and $y = \sqrt{x}$.

   A. 1
   B. 2
   C. 0
   D. 6

(d) (2 points) The Jacobian of the function $G(u, v, w) = (\sin u, u^2 + v^2, \ln(vw))$

A. $\frac{2v \cos u}{u}$
B. $\frac{u \cos v}{w}$
C. $\frac{2u \cos u}{v}$
D. $\frac{2v \cos u}{w}$

(e) (2 points) Integrate $\sqrt{x^2 + y^2}$ over the ball $x^2 + y^2 + z^2 \leq 4$.

A. $\frac{4\pi^2}{3}$
B. $4\pi^2$
C. $\frac{4}{3}$
D. $\frac{4\pi}{3}$

(f) (2 points) Calculate the line integral of $f(x, y) = 1$ along the triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$.

A. $2 + \sqrt{2}$**.**
B. $2 - \sqrt{2}$.
C. 2.
D. -2.

2. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

   (a) (2 points) Suppose that $\mathbf{F}$ is a vector field such that $\mathbf{F} \cdot \langle x, y\rangle > 0$ for all $(x, y) \neq (0, 0)$. Let $\mathcal{C}$ be the unit circle oriented counter clockwise. The flux through $\mathcal{C}$ of the vector field $\mathbf{F}$ is

   A. greater than and sometimes equal to zero.
   B. less than and sometimes equal to zero.
   C. always greater than zero.
   D. always less than zero.

   (b) (2 points) The vector field $r^{-4}\langle x, y\rangle$ where $r^2 = x^2 + y^2$, has domain $\mathbb{R}^2 - \{(0, 0)\}$. The vector field

   A. has zero curl and is conservative.
   B. has non zero curl and is conservative.
   C. has zero curl and is not conservative.
   D. has non zero curl and is not conservative.

   (c) (2 points) Suppose $\varphi(x, y)$ is a scalar function defined on $\mathbb{R}^2$ with the property that $\varphi(-3, 0) = 5$ and $\varphi(3, 0) = 3$. Let $\mathbf{F} = \nabla\varphi$. What is the value of $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ if $\mathcal{C}$ is the top half of the ellipse

$$\frac{x^2}{9} + \frac{y^2}{2} = 1$$

   oriented counter clockwise.

   A. Not enough information.
   B. 0.
   C. -2.
   D. 2.

(d) (2 points) Consider the surface $\mathcal{S}$ parametrised by $G(u, v) = (v^2, uv, u + v)$. Which of these vectors is tangent to the surface at the point $(1, 1, 2)$?

A. $\langle 2, -1, 1\rangle$
B. $\langle 1, -1, 1\rangle$.
C. $\langle 1, 1, 0\rangle$.
D. $\langle 4, -1, -1\rangle$**.**

(e) (2 points) Suppose $\mathbf{F}$ has $\mathrm{div}(\mathbf{F}) = 1$. What is the (outward) flux of $\mathbf{F}$ through the sphere

$$x^2 + y^2 + z^2 = 4?$$

A. $32\pi/3$**.**
B. $4\pi/3$.
C. $-\pi/3$.
D. $12\pi/3$.

(f) (2 points) Let $\mathcal{C}_1$ be the unit circle, and $\mathcal{C}_2$ the circle of radius 2, both oriented counter clockwise, and centered at the origin. Let $\mathcal{D}$ be annulus between these circles. If $\mathbf{F} = \langle x - y, xy\rangle$, the integral $\oint_{\mathcal{C}_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is equal to

A. $\iint_{\mathcal{D}} y + 1 \, \mathrm{d}S + \oint_{\mathcal{C}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$
B. $\iint_{\mathcal{D}} y - 1 \, \mathrm{d}S - \oint_{\mathcal{C}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$
C. $-\iint_{\mathcal{D}} y + 1 \, \mathrm{d}S + \oint_{\mathcal{C}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$
D. $-\iint_{\mathcal{D}} y + 1 \, \mathrm{d}S - \oint_{\mathcal{C}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$

3. Consider the two vector fields

$$\mathbf{F}(x, y, z) = \langle x, \frac{-z}{y^2 + z^2}, \frac{y}{y^2 + z^2} \rangle \quad \text{and} \quad \mathbf{H}(x, y, z) = \langle x, \frac{y}{y^2 + z^2}, \frac{z}{y^2 + z^2} \rangle.$$

(a) (2 points) Which of these vector fields is conservative on the domain $\mathbb{R}^3 - \{(x, 0, 0) \mid x \in \mathbb{R}\}$? *Hint: think about the next two parts first.*

Solution: We might have a feeling that **F** is not conservative since it looks a bit like the vortex vector field in the second two coordinates, and this is correct! **H** is conservative since it has a potential function (see the next part).

(b) (4 points) For the conservative vector field, find a potential function.

Solution: Let f be a potential function for **H**, then

$$\begin{aligned}
\partial_x f &= x & &\text{so } f = \frac{1}{2}x^2 + \alpha(y, z) \\
\partial_y f &= \frac{y}{y^2 + z^2} & &\text{so } f = \frac{1}{2} \ln \left| y^2 + z^2 \right| + \beta(x, z) \\
\partial_z f &= \frac{z}{y^2 + z^2} & &\text{so } f = \frac{1}{2} \ln \left| y^2 + z^2 \right| + \gamma(x, y)
\end{aligned}$$

Thus a potential function is $f = \frac{1}{2} \ln \left| y^2 + z^2 \right| + \frac{1}{2}x^2$.

(c) (4 points) Show that the other vector field is not conservative.

Solution: Since $\mathbf{F}$ looks like the vortex vector field in the second two coordinates, we might get the hint that we should calculate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ where $\mathcal{C}$ is the unit circle in the yz-plane oriented counter clockwise when looking from the positive x direction. This is parametrised by

$$\mathbf{r}(t) = (0, \cos t, \sin t) \quad \text{where} \quad t \in [0, 2\pi].$$

And the tangent is given by

$$\mathbf{r}'(t) = \rangle 0, -\sin t, \cos t \langle$$

so we see that $\mathbf{F}(\mathbf{r}(t)) = (0, -\sin t, \cos t)$ and hence the integral is

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}''(t)\, dt = \int_0^{2\pi} 1\, dt = 2\pi \neq 0.$$

Thus the vector field cannot be conservative.

(d) (2 points) Calculate line integral of the non conservative vector field along the oriented curve $\mathcal{C}$ given by the parametrisation

$$\mathbf{r}(t) = (\cos t, 2\sin t, \cos t)\,, \text{ where } t \in [0, 2\pi].$$

Solution: We could go ahead an compute this direcly. However there is an easier way. We could also, with a little work, describe exactly what this curve is (see below) but we don't need to. We just need to understand the rough behaviour of this curve.

When $t = 0$ we are at the point $(1, 0, 1)$. As we increase t, the y-coord gets bigger, and the x and z-coords get smaller, until be get to $t = \pi/2$ and the point $(0, 2, 0)$. We keep going, with x and z becoming negative and y decreasing to zero, until we get to $(-1, 0, -1)$ when $t = \pi$. We keep going and loop back through the $y < 0$ region back to our starting point.

This description and the fact that $\nabla \times \mathbf{F} = 0$ suggests we should restrict $\mathbf{F}$ to the domains $z \geq 0$ and $z \leq 0$. Both are simply connected and thus $\mathbf{F}$ is conservative on them. Now we see that instead of taking the loop given in the question, we could simply take the unit circle of radius one, in the yz-plane, centred at the origin but going clockwise. By the previous question, this is -2π.

The curve happens to be an ellipse, centred at the origin, lying in the plane $x = z$.

4. (a) (3 points) Let $\mathcal{S}$ be an oriented surface and let $\mathbf{F}$ be a vector field which has continuous partial derivatives on an open region containing $\mathcal{S}$. State Stokes' theorem for $\mathbf{F}$.

Solution: If ∂S is the boundary of S equipped with the boundary orientation then

$$\iint_{\mathcal{S}} \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

(b) (3 points) Let $\mathcal{S}$ be a closed surface that encloses a solid region $\mathcal{E}$ in R^3. Assume that $\mathcal{S}$ is piecewise smooth and is oriented with normal vectors pointing to the outside of $\mathcal{E}$. Let $\mathbf{F}$ be a vector field whose domain contains $\mathcal{E}$. State the divergence theorem for $\mathbf{F}$.

Solution:

$$\iiint_{\mathcal{E}} \nabla \cdot \mathbf{F} \, dV = \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$$

(c) (3 points) Consider the vector field $\mathbf{F} = \rho^{-3}\langle x, y, (1+\rho^3)z\rangle$ where $\rho = \sqrt{x^2+y^2+z^2}$. What does the divergence theorem predict the outward flux of $\mathbf{F}$ through the unit sphere $x^2+y^2+z^2 = 1$ to be?

Solution: The divergence theorem says we should be able to calculate this using a triple integral. First we take the divergence: note that $\partial_x \rho^{-3} = -3x\rho^{-5}$ and similarly for y and z so

$$\begin{aligned}\nabla\cdot\mathbf{F} &= -3x^2\rho^{-5} + \rho^{-3} - 3y^2\rho^{-5} + \rho^{-3} - 3z^2\rho^{-5} + \rho^{-3} + 1\\ &= -3\rho^{-5}(x^2+y^2+z^2) + 3\rho^{-3} + 1 = 1\end{aligned}$$

Thus the flux is equal to

$$\iiint_{\mathcal{E}} 1\ dV$$

where $\mathcal{E}$ is the unit ball. This is just the volume of the unit ball which is $4\pi/3$.

(d) (3 points) What is the outward flux of $\mathbf{F}$ through the unit sphere $x^2+y^2+z^2 = 1$?

Solution: Lets now calculate the flux properly. On the sphere we have $\rho = 1$ so we have $\mathbf{F} = \langle x, y, 2z\rangle$. We can parametrise the surface by

$$G(\theta,\phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$$

The tangent and normal vectors are thus

$$\begin{aligned}\mathbf{T}_\theta &= \langle -\sin\theta\sin\phi, \cos\theta\sin\phi, 0\rangle\\ \mathbf{T}_\phi &= \langle \cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi\rangle\\ \mathbf{N} &= \langle -\cos\theta\sin^2\phi, -\sin\theta\sin^2\phi, -\sin\phi\cos\phi\rangle\end{aligned}$$

We can see that this normal vector (say at $\theta = 0$ and $\phi = \pi/2$) is pointing inwards, so we have accidentally parametrised $-\mathcal{S}$. Thats ok since we now have

$$\begin{aligned}\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} &= -\iint_{-\mathcal{S}} \mathbf{F}\cdot d\mathbf{S}\\ &= -\int_0^{2\pi}\int_0^{\pi} -\cos^2\theta\sin^3\phi - \sin^2\theta\sin^3\phi - 2\sin\phi\cos^2\phi\ d\phi\ d\theta\\ &= \int_0^{2\pi}\int_0^{\pi} \sin^3\phi + 2\sin\phi\cos^2\phi\ d\phi\ d\theta\\ &= 2\pi\int_0^{\pi} \sin^3\phi + 2\sin\phi\cos^2\phi\ d\phi\\ &= 2\pi\left(\frac{4}{3} + \left[-\frac{2}{3}\cos^3\phi\right]_0^{\pi}\right)\\ &= 2\pi\left(\frac{4}{3} + \frac{2}{3} + \frac{2}{3}\right) = \frac{16\pi}{3}\end{aligned}$$

(e) (1 point) Why do these two values not agree?

Solution: $\mathbf{F}$ is not defined at the origin!

5. $\mathcal{S}$ be the hyperboloid $x^2 + y^2 - z^2 = 1$ where $0 \le z \le a$ with orientation given by normal pointing away from the origin.

   (a) (3 points) Find the volume enclosed by $\mathcal{S}$ and the planes $z = a$ and $z = 0$.

Solution: We should be able to do this directly. There are two obvious ways to describe the region. The first is as

$$0 \le z \le a, \text{ and } (x, y) \in \mathcal{D}_z = \text{ the disk of radius } \sqrt{1 + z^2}$$

Thus the volume is given by

$$\iiint_{\mathcal{E}} 1 \, dV = \int_0^a \iint_{\mathcal{D}_z} 1 \, dA \, dz = \int_0^a \pi(1 + z^2) \, dz = a\pi \left(1 + \frac{1}{3}a^2\right)$$

Where we have used the fact that $\iint_{\mathcal{D}_z} 1 \, dA$ is just the area of this disk.

We can also use a different method (this is the original method used in these solutions, but I made an error and in any case it is a little more complicated than the above, so I'm just including it for completeness).

We can describe the region as z-simple by

$$f(x, y) \le z \le a \quad \text{and} \quad (x, y) \in \mathcal{D}$$

where

$$f(x, y) = \begin{cases} \sqrt{x^2 + y^2 - 1} & \text{if } x^2 + y^2 \ge 1, \\ 0 & \text{if } x^2 + y^2 \le 1. \end{cases}$$

and where $\mathcal{D}$ is the disk of radius $\sqrt{1 + a^2}$. Let $\mathcal{A}$ be the annulus of inner radius 1 and outer radius $\sqrt{1 + a^2}$. Thus (using polar coords to integrate over the region $\mathcal{A}$ and using the fact that $a\pi$ is the volume of the cylinder over the unit disk),

$$\begin{aligned}
\iiint_{\mathcal{E}} 1 \, dV &= \iint_{\mathcal{D}} \int_{f(x,y)}^a 1 \, dz \, dA \\
&= \iint_{\mathcal{A}} \int_{\sqrt{x^2+y^2-1}}^a 1 \, dz \, dA + \pi a \\
&= \int_0^{2\pi} \int_1^{\sqrt{1+a^2}} r \int_{\sqrt{r^2-1}}^a 1 \, dz \, dr \, d\theta + \pi a \\
&= 2\pi \int_1^{\sqrt{1+a^2}} ar - r\sqrt{r^2 - 1} \, dr + \pi a \\
&= 2\pi \left[\frac{1}{2}ar^2 - \frac{1}{3}(r^2 - 1)^{3/2}\right]_1^{\sqrt{1+a^2}} + \pi a \\
&= a\pi \left(1 + \frac{1}{3}a^2\right).
\end{aligned}$$

   (b) (2 points) Find a parametrisation $G(\theta, z)$ for $\mathcal{S}$ such that $(\theta, z) \in [0, 2\pi] \times [0, a]$.

Solution:

$$G(\theta, z) = \left(\sqrt{z^2 + 1} \cos\theta, \sqrt{z^2 + 1} \sin\theta, z\right)$$

(c) (5 points) Calculate the flux of the vector field $\mathbf{F} = \langle 1, 1, -2\rangle$ through $\mathcal{S}$.

Solution: The tangent and normal vectors are thus

$$\begin{aligned}
\mathbf{T}_\theta &= \langle -\sqrt{1+z^2}\sin\theta, \sqrt{z^2+1}\cos\theta, 0\rangle \\
\mathbf{T}_z &= \langle \frac{z\cos\theta}{\sqrt{1+z^2}}, \frac{z\sin\theta}{\sqrt{1+z^2}}, 1\rangle \\
\mathbf{N} &= \langle \sqrt{z^2+1}\cos\theta, \sqrt{z^2+1}\sin\theta, -z\rangle
\end{aligned}$$

So

$$\begin{aligned}
\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^a \langle 1,1,-2\rangle \cdot \langle \sqrt{z^2+1}\cos\theta, \sqrt{z^2+1}\sin\theta, -z\rangle \; dz\; d\theta \\
&= \int_0^{2\pi}\int_0^a \sqrt{z^2+1}(\cos\theta+\sin\theta) + 2z \; dz\; d\theta \\
&= \left(\int_0^{2\pi} \cos\theta + \sin\theta \; d\theta\right)\left(\int_0^a \sqrt{z^2+1}\; dz\right) + 2\pi\int_0^a 2z\; dz \\
&= 0\cdot\left(\int_0^a \sqrt{z^2+1}\; dz\right) + 2\pi a^2 = 2\pi a^2.
\end{aligned}$$

6. Consider an oriented curve, $\mathcal{C}$ in $\mathbb{R}^2$, which consists of straight lines between the points

$$(0,0),(2,-1),(3,2),(1,3) \text{ and back to } (0,0)$$

in that order. Notice that this path forms a parallelogram.

(a) (3 points) Find a change of variables $G(u,v)$ which maps the unit square $[0,1]\times[0,1]$ to this parallelogram. Calculate the Jacobian of $G(u,v)$.

Solution:

$$G(u,v) = (2u+v, 3v-u) \quad \text{and} \quad J(G) = 7$$

(b) (4 points) Let $\mathbf{F} = \langle -xy, y^2\rangle$. Use Green's theorem to calculate $\oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$. *To receive full credit you must use Green's theorem.*

Solution:

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} &= \iint_{\mathcal{D}} \mathrm{curl}(\mathbf{F})\; dA\\ &= \iint_{\mathcal{D}} x\; dA\\ &= \int_0^1\int_0^1 (2u+v)\cdot 7\; du\; dv = \frac{21}{2}.\end{aligned}$$

(c) (4 points) Use Green's theorem to calculate the *flux* of $\mathbf{F}$ through $\mathcal{C}$. *To receive full credit you must use Green's theorem.*

Solution: We use the Flux version of Greens theorem.

$$\begin{aligned}
\int_{\mathcal{C}} (\mathbf{F} \cdot \mathbf{n}) ds &= \iint_{\mathcal{D}} \operatorname{div}(\mathbf{F}) \, dA \\
&= \iint_{\mathcal{D}} y \, dA \\
&= \int_0^1 \int_0^1 (3v - u) \cdot 7 \, du \, dv = 7.
\end{aligned}$$

7. Consider the vector field $\mathbf{F} = \text{curl}(\mathbf{A})$ where $\mathbf{A} = \langle z(x+y) - y^2, x(z-y), x^2+y^2 \rangle$.

   (a) (3 points) Calculate $\mathbf{F}$.

   Solution:
   $$\mathbf{F} = \langle 2y - x, y - x, y \rangle$$

   (b) (7 points) Suppose $\mathcal{S}$ is an oriented, closed surface and consider its intersection with the plane $ax + by + cz = 1$. The plane splits the surface into two halves $\mathcal{S}_+$ and $\mathcal{S}_-$ (e.g. the unit sphere is split into two pieces by the plane $z = \frac{1}{2}$). These are both surfaces with boundary. Find conditions on a, b, and c such that the flux of $\mathbf{F}$ through both $\mathcal{S}_+$ and $\mathcal{S}_-$ is zero for any choice of surface $\mathcal{S}$. *Full credit will only be given if your answer states clearly each fact/theorem you are using and determines an infinite number of possibles values for a, b, and c.*

   Solution: Let $\mathcal{D}$ be the region in the plane bounded by $\partial\mathcal{S}_+$ with orientation given by a normal vector determined by the orientation of ∂S_+. Since $\mathbf{F} = \text{curl}(\mathbf{A})$, surface independence implies that
   $$\iint_{\mathcal{S}_+} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{D}} \mathbf{F} \cdot d\mathbf{S}$$
   and similarly, after taking orientations into account,
   $$\iint_{\mathcal{S}_-} \mathbf{F} \cdot d\mathbf{S} = -\iint_{\mathcal{D}} \mathbf{F} \cdot d\mathbf{S}.$$
   We know that the integral over $\mathcal{D}$ will be zero if $\mathbf{F}$ is perpendicular to the normal vector of the plane which is $\langle a, b, c \rangle$. I.e.
   $$\begin{aligned} 0 &= \langle 2y - x, y - x, y \rangle \cdot \langle a, b, c \rangle \\ &= -(a+b)x + (2a + by + c)y \end{aligned}$$
   for any x, y. By comparing coefficients we see that $a + b = 0$ and $2a + b + c = 0$. We can summarise this as saying that as long as
   $$a = t, b = -t, c = -t, \quad \text{for some } t \in \mathbb{R},$$
   the integrals over the two halves will be zero.

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