Worksheet 9

(1) Consider the vector field $\boldsymbol{F} = \langle -y, x, z \rangle$.

(a) Compute $\operatorname{curl}(\boldsymbol{F})$.

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -y & x & z \end{vmatrix} = \langle 0, 0, 2 \rangle .$$

(b) For the surface $S : z = 1 - x^2 - y^2$ above the xy-plane compute $\iint_S \operatorname{curl}(\boldsymbol{F}) \cdot d\boldsymbol{S}$ directly (i.e. by parametrizing the surface).

$$G(r,\theta) = (r\cos\theta,\ r\sin\theta,\ 1 - r^2) \qquad 0 \le \theta \le 2\pi$$
$$0 \le r \le 1$$

$$G_r = (\cos\theta, \sin\theta, -2r)$$

$$G_\theta = (-r\sin\theta, r\cos\theta, 0)$$

$$G_r \times G_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle -2r^2\cos\theta,\ 2r^2\sin\theta,\ r \rangle$$

Assuming upwards orientation, this is then the normal.

$$\iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S} = \int_0^{2\pi} \int_0^1 2r \, dr \, d\theta$$

$$= 2\pi \cdot 2 \cdot \frac{1}{2} = 2\pi$$

(c) Check your answer to part (b) using Stokes' theorem (i.e. compute a line integral).

The boundary is the oriented by $r(t) = \langle \cos t, \sin t, 0 \rangle$.

$$r'(t) = \langle -\sin t, \cos t, 0 \rangle$$

$$F(r(t)) = \langle -\sin t, \cos t, 0 \rangle .$$

$F(r(t)) = \langle -\sin t, \cos t, 0 \rangle.$

$$\iint_S \operatorname{curl}\vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot ds$$

$$= \int_0^{2\pi} \langle -\sin t, \cos t, 0 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt$$

$$= 2\pi.$$

(2) Let C be a simple closed curve in the plane $x+y+z=1$. Show that for $\boldsymbol{F} = \langle az, bx, cy \rangle$ the integral

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$$

only depends on the area of the region enclosed by C, and not on its shape or location in the plane.

$$\operatorname{curl}\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ az & bx & cy \end{vmatrix} = \langle c, a, b \rangle$$

Let D be the region enclosed by C on the plane. Note that the plane has unit normal $\vec{n} = \frac{1}{\sqrt{3}}(1,1,1)$.

Hence, $$\int_C \vec{F} \cdot dr = \iint_D \operatorname{curl}\vec{F} \cdot d\vec{S}$$

$$= \iint_D \operatorname{curl}\vec{F} \cdot \vec{n} \, dS$$

$$= \frac{1}{\sqrt{3}}(a+b+c) \iint_D dS$$

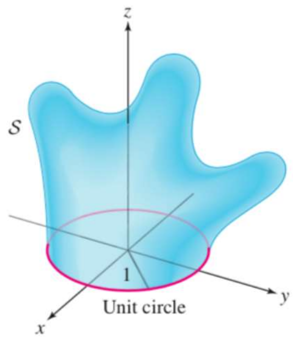
$$= \frac{1}{\sqrt{3}}(a+b+c) \operatorname{Area}(D).$$

(3) You know only two things about a vector field $\boldsymbol{F}$:

(a) $\boldsymbol{F}$ has a vector potential $\boldsymbol{A}$ (this means that $\boldsymbol{F} = \text{curl}\, A$).

(b) the line integral of $\boldsymbol{A}$ around the unit circle (oriented counterclockwise) in the xy-plane is 25.

Determine the flux of $\boldsymbol{F}$ through the surface $\boldsymbol{S}$ below, oriented with an upward-pointing normal.



$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \text{curl}\, \vec{A} \cdot d\vec{S}$$

$$= \int_C \vec{A} \cdot dr$$ by Stokes thm (note orientations agree)

$$= 25.$$