Worksheet 8

(1) Use Green's Theorem to evaluate the integral $\oint_C e^{2x+y}\,dx + e^{-y}\,dy$, where C is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$, oriented **clockwise**.

Let D be the region enclosed by the triangle. Then green's thm gives

$$\oint_C e^{2x+y}dx + e^{-y}dy = -\iint_D 0 - e^{2x+y}\,dx\,dy \quad \text{(negative as oriented clockwise)}$$

$$= \int_0^1\int_0^x e^{2x+y}\,dy\,dx$$

$$= \int_0^1 e^{2x+y}\Big|_0^x\,dx$$

$$= \int_0^1 e^{3x} - e^{2x}\,dx$$

$$= \frac{1}{3}e^{3x} - \frac{1}{2}e^{2x}\Big|_0^1$$

$$= \frac{1}{3}e^3 - \frac{1}{2}e^2 - \frac{1}{3} + \frac{1}{2}$$

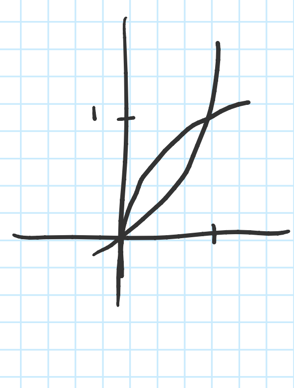
$$= \frac{1}{3}e^3 - \frac{1}{2}e^2 + \frac{1}{6}$$

(2) Use Green's Theorem to evaluate $\oint_C (y + e^{\sqrt{x}})\,dx + (x^4 + 2x^2y^2)\,dy$, where C is the boundary of the region enclosed by the parabolas $y = x^2$, $x = y^2$, oriented counterclockwise.

$$\oint_C (y+e^{\sqrt{x}})\,dx + (x^4+2x^2y^2)\,dy$$

$$= \iint_D 4x^3 + 4xy^2 - 1\,dx\,dy$$

$$= \int_0^1\int_{y^2}^{\sqrt{y}} 4x^3 + 4xy^2 - 1\,dx\,dy$$



$$= \int_0^1 \int_{y^2}^{\sqrt{y}} 4x^3 + 4xy^2 - 1 \, dx\, dy$$

$$= \int_0^1 x^4 + 2x^2y^2 - x \Big|_{y^2}^{\sqrt{y}} dy$$

$$= \int_0^1 y^2 - y^8 + 2y^3 - 2y^6 - \sqrt{y} + y^2 \, dy$$

$$= \frac{2}{3}y^3 - \frac{1}{9}y^9 + \frac{1}{2}y^4 - \frac{2}{7}y^7 - \frac{2}{3}y^{3/2} \Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{9} + \frac{1}{2} - \frac{2}{7} - \frac{2}{3}$$

= something

(3) (a) By parametrizing the line segment $\mathcal{C}_1$ from (x_1, y_1) to (x_2, y_2), show that $\int_{\mathcal{C}_1} x\,dy - y\,dx = x_1y_2 - x_2y_1$.

(b) Suppose that the vertices of a triangle are (x_1, y_1), (x_2, y_2), and (x_3, y_3), in counterclockwise order. Show that the area of the triangle is

$$A = \frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3).$$

Hint: if $\mathcal{C}$ is the boundary of the triangle, show that $\int_{\mathcal{C}} x\,dy - y\,dx$ is twice its area

(c) Use the same idea as in part (b) to find the area of the pentagon with vertices $(0, 0)$, $(2, 1)$, $(1, 3)$, $(0, 2)$, and $(-1, 1)$.

(a) $r(t) = (1-t)(x_1, y_1) + t(x_2, y_2) = (x_1 + t(x_2 - x_1),\ y_1 + t(y_2 - y_1))$

$r'(t) = (x_2 - x_1,\ y_2 - y_1)$

Hence $\int_C x\,dy - y\,dx = \int_0^1 (x_1 + t(x_2 - x_1))(y_2 - y_1) - (y_1 + t(y_2 - y_1))(x_2 - x_1)\,dt$

$$= \int_0^1 x_1(y_2 - y_1) - y_1(x_2 - x_1)\,dt$$

$$= \int_0^1 x_1y_2 - x_2y_1\,dt$$

$$= x_1y_2 - x_2y_1$$

(b) By Green's

$$\text{Area} = \iint_D dA = \frac{1}{2}\int_{C_1+C_2+C_3} x\,dy - y\,dx$$

$$= \frac{1}{2}\left(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3\right)$$

(c) $(0,0) \to (2,1)$ $\quad \int_C I = 0$ where $I = x\,dy - y\,dx$

$(2,1) \to (1,3)$ $\quad \int_C I = 2\cdot 3 - 1\cdot 1 = 5$

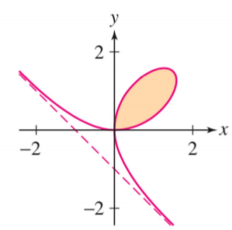
$(1,3) \to (0,2)$ $\quad \int_C I = 2 - 0 = 2$

$(0,2) \to (-1,1)$ $\quad$ '' $= 0 - 2$

$(-1,1) \to (0,0)$ $\quad$ '' $= 0.$

Hence $\text{Area} = \frac{1}{2}(5 + 2 - 2) = \frac{5}{2}$

(4) The *folium of Descartes* is the graph of $x^3 + y^3 = 3xy$ (see right).



(a) We can parametrize the folium by

$$x(t) = \frac{3t}{1+t^3}, \qquad y(t) = \frac{3t^2}{1+t^3}, \qquad -\infty < t < -1 \text{ and } -1 < t < \infty.$$

Show that

$$x(t)y'(t) - y(t)x'(t) = \frac{9t^2}{(1+t^3)^2}.$$

Hint: you're welcome to just compute this directly and simplify, but if you want a clever way to do it, use that the quotient rule says

$$x(t)^2 \frac{d}{dt}\left(\frac{y(t)}{x(t)}\right) = x(t)y'(t) - y(t)x'(t).$$

(b) Find the area of the loop of the folium. (The boundary of the loop is the part of the parametrization corresponding to $0 \le t < \infty$).

(a) $x(t) = \dfrac{3t}{1+t^3}$ $\qquad x'(t) = \dfrac{3(1+t^3) - 9t^3}{(1+t^3)^2} = \dfrac{3 - 6t^3}{(1+t^3)^2}$

$$y(t) = \frac{3t^2}{1+t^3} \qquad y'(t) = \frac{6t(1+t^3) - 9t^4}{(1+t^3)^2} = \frac{6t - 3t^4}{(1+t^3)^2}$$

$$x(t)y'(t) - y(t)x'(t) = \frac{3t(6t-3t^4) - 3t^2(3-6t^3)}{(1+t^3)^3}$$

$$= \frac{18t^2 - 9t^5 - 9t^2 + 18t^5}{(1+t^3)^3}$$

$$= \frac{9t^2 + 9t^5}{(1+t^3)^3}$$

$$= \frac{9t^2}{(1+t^3)^2}$$

Also, the clever way:

$$\frac{y(t)}{x(t)} = t, \quad \text{so} \quad x(t)^2 \frac{d}{dt}\left(\frac{y(t)}{x(t)}\right) = x(t)^2 = \frac{9t^2}{(1+t^3)^2}$$

Hence $xy' - yx' = \dfrac{9t^2}{(1+t^3)^2}$.

(b) $$\text{Area} = \frac{1}{2}\int_C x\,dy - y\,dx = \frac{1}{2}\int_0^\infty \frac{9t^2}{(1+t^3)^2}\,dt$$

$$= -\frac{3}{2}\frac{1}{(1+t^3)}\Big|_0^\infty = \frac{3}{2}$$