Worksheet 7

(1) The gravitational potential V at P due to a point mass m sitting at the point Q is given by

$$V(P) = -\frac{Gm}{r}, \tag{1}$$

where $r = \|P - Q\|$ is the distance from P to Q and G is the gravitational constant.

Suppose that instead of a point mass, we have a thin surface $\mathcal{S}$ with mass density $\delta(x, y, z)$; then the gravitational potential is given by

$$V(P) = -G \iint_{\mathcal{S}} \frac{\delta(x,y,z)\, dS}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}},$$

where $P = (a, b, c)$.

(a) Suppose the surface $\mathcal{S}$ is a hollow sphere of radius R centered at the origin with mass density $\delta = m/(4\pi R^2)$. By symmetry, we only need to compute $V(P)$ at a single point $P = (0, 0, r)$ with $r \neq R$. Use spherical coordinates to show that

$$V(0,0,r) = -\frac{Gm}{4\pi} \int_0^{\pi} \int_0^{2\pi} \frac{\sin\phi \, d\theta \, d\phi}{\sqrt{R^2 + r^2 - 2Rr\cos\phi}}.$$

I'm lazy to derive this, so from textbook:

- Sphere of radius R, centered at the origin:

  $G(\theta, \phi) = (R\cos\theta\sin\phi,\ R\sin\theta\sin\phi,\ R\cos\phi)$

  Unit radial vector: $\mathbf{e}_r = \langle \cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi \rangle$

  Outward normal: $\mathbf{N} = \mathbf{T}_\phi \times \mathbf{T}_\theta = (R^2\sin\phi)\,\mathbf{e}_r$

  $dS = \|\mathbf{N}\|\, d\phi\, d\theta = R^2 \sin\phi\, d\phi\, d\theta$

Hence $V(0,0,r) = -G \iint_S \dfrac{m/(4\pi R)^2}{(x^2 + y^2 + (z-r)^2)^{1/2}}\, dS$

$$= -\frac{Gm}{(4\pi R)^2} \int_0^{2\pi} \int_0^{\pi} \frac{R^2 \sin\phi\, d\phi\, d\theta}{(R^2\cos^2\theta\sin^2\phi + R^2\sin^2\theta\sin^2\phi + (R\cos\phi - r)^2)^{1/2}}$$

$$= -\frac{Gm}{(4\pi R)^2} \int_0^{2\pi} \int_0^{\pi} \frac{R^2 \sin\phi\, d\phi\, d\theta}{(R^2\sin^2\phi + R^2\cos^2\phi - 2rR\cos\phi + r^2)^{1/2}}$$

$$= \frac{-Gm}{\ldots} \int^{2\pi} \int^{\pi} \frac{\sin\phi\, d\phi\, d\theta}{\ldots}$$

$$= \frac{-Gm}{(4\pi)^2} \int_0^{2\pi} \int_0^{\pi} \frac{\sin\phi \, d\phi \, d\theta}{\sqrt{R^2 + r^2 - 2Rr\cos\phi}}$$

(b) To evaluate this integral, it's helpful to use the substitution $u = R^2 + r^2 - 2Rr\cos\phi$. Show that when $\phi = 0$, $u = (R-r)^2$, and when $\phi = \pi$, $u = (R+r)^2$. Use this to show that

$$V(0,0,r) = -\frac{Gm}{2Rr}\left(|R+r| - |R-r|\right).$$

when $\phi = 0$, $u = R^2 + r^2 - 2Rr = (R-r)^2$

$\phi = \pi$, $u = R^2 + r^2 + 2Rr = (R+r)^2$.

$du = 2Rr\sin\phi \, d\phi$

Hence, $$V(0,0,r) = \frac{-Gm}{8Rr\pi} \int_0^{2\pi} \int_{(R-r)^2}^{(R+r)^2} \frac{du}{\sqrt{u}} \, d\theta$$

$$= \frac{-Gm}{4Rr} \int_{(R-r)^2}^{(R+r)^2} \frac{du}{\sqrt{u}}$$

$$= -\frac{Gm}{2Rr}\left(|R+r| - |R-r|\right)$$

(c) Suppose first that P is outside the sphere (so $r > R$). Show that $V(P) = -\frac{Gm}{r}$.

If $r > R$, then $|R+r| = R+r$ and $|R-r| = r-R$.

Hence $$V(P) = \frac{-Gm}{2Rr}\left(R+r-(r-R)\right)$$

$$= -\frac{Gm}{r}$$

(d) If P is inside the sphere (so $r < R$) show that $V(P) = -\dfrac{Gm}{R}$.

if $r < R$, then $|R+r| = R+r$, $|R-r| = R-r$.

Hence $V(P) = -\dfrac{Gm}{2Rr}\left(R+r-(R-r)\right)$

$$= -\frac{Gm}{R}$$

(e) Let's interpret this result (keep in mind that R is a constant and r is a variable). If the object is inside the sphere, then the sphere exerts no gravitational force on the object (remember that $\boldsymbol{F} = -\nabla V$). On the other hand, if the object is outside the sphere, then the sphere behaves like a point mass as far as gravity is concerned (look at equation (1)). Newton was the first to prove this somewhat surprising fact.

(2) Let $\mathcal{S}$ be the ellipsoid $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{z}{2}\right)^2 = 1$. Calculate the flux of $\boldsymbol{F} = z\boldsymbol{i}$ over the portion of $\mathcal{S}$ where $x, y, z \geq 0$ with upward-pointing normal.
Hint: parametrize $\mathcal{S}$ using a modified form of spherical coordinates.

parameterisation: $G(s,t) = (4\cos s \sin t, 3\sin s \sin t, 2\cos t)$

$G_s(s,t) = (-4\sin s \sin t, 3\cos s \sin t, 0)$

$G_t(s,t) = (4\cos s \cos t, 3\sin s \cos t, -2\sin t)$

$s \in [0, \pi/2]$, $t \in [0, \frac{\pi}{2}]$

$$G_s(s,t) \times G_t(s,t) = \begin{vmatrix} i & j & k \\ -4\sin s \sin t & 3\cos s \sin t & 0 \\ 4\cos s \cos t & 3\sin s \cos t & -2\sin t \end{vmatrix}$$

$$N(s,t) = \langle -6\cos s \sin^2 t, -8\sin s \sin^2 t, -12\sin t \cos t \rangle$$

Test a point to see if outward facing: $t=0$, $s=\frac{\pi}{2}$

$N(0,0) = \langle 0, 0, -12 \rangle$ so inward. Hence swap signs.

$$N(s,t) = \langle 6\cos s \sin^2 t,\ 8\sin s \sin^2 t,\ 12\sin t\cos t \rangle.$$

$$F(G(s,t)) = \langle 2\cos t, 0, 0 \rangle$$

$$F(G(s,t)) \cdot N(s,t) = 12\cos s \sin^2 t \cos t$$

So integral is:

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^{\pi/2}\int_0^{\pi/2} 12\cos s \sin^2 t \cos t \, ds\, dt$$

$$= 12\left(\int_0^{\pi/2} \cos s\, ds\right)\left(\int_0^{\pi/2} \sin^2 t \cos t\, dt\right)$$

Now, $\displaystyle\int_0^{\pi/2} \cos s\, ds = \sin s \Big|_0^{\pi/2} = 1$

$$\int_0^{\pi/2} \sin^2 t \cos t\, dt = \frac{\sin^3 t}{3}\Big|_0^{\pi/2} = \frac{1}{3}$$

Hence, $\displaystyle\iint_S \vec{F} \cdot d\vec{S} = 4.$