Worksheet 6

(1) Let $\boldsymbol{F}$ be the vortex field

$$\boldsymbol{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle.$$

(a) Let $\mathcal{C}$ be the straight line segment from $P = (a, b)$ to $Q = (a, c)$, where $a > 0$ and $b < c$. Show that

$$\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r}$$

is equal to the angle between the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$.

Method 1: The parameterisation is given by $r(t) = (1-t)(a,b) + t(a,c)$

$\Rightarrow r(t) = (a, (1-t)b + tc)$

so $r'(t) = \langle 0, c-b \rangle$

$$\vec{F}(r(t)) = \left(\frac{-(1-t)b+tc}{a^2+((1-t)b+tc)^2}, \frac{a}{a^2+((1-t)b+tc)^2} \right)$$

$$\vec{F}(r(t)) \cdot r'(t) = \frac{a(c-b)}{a^2+((1-t)b+tc)^2} = \frac{a(c-b)}{a^2+(b+(c-b)t)^2}$$

$$I = \int_0^1 \vec{F}(r(t)) \cdot r'(t)\,dt = \frac{1}{a} \int_0^1 \frac{c-b}{1+\left(\frac{b+(c-b)t}{a}\right)^2}\,dt$$

let $u = \frac{b+(c-b)t}{a}$, $du = \frac{c-b}{a}\,dt$ when $t=0$, $u = \frac{b}{a}$

$t=1$, $u = \frac{c}{a}$

$$\therefore I = \int_{b/a}^{c/a} \frac{du}{1+u^2} = \arctan\left(\frac{c}{a}\right) - \arctan\left(\frac{b}{a}\right) = \angle OQ - \angle OP.$$

Method 2: (polar) let θ_1, θ_2 be angle of $\overrightarrow{OP}, \overrightarrow{OQ}$ respectively.

In polar coordinates the line is given by $r = a\sec\theta$ for $\theta \in [\theta_1, \theta_2]$.

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$$r(\theta) = (f(\theta)\cos\theta, f(\theta)\sin\theta) = (a\sec\theta\cos\theta, a\sec\theta\sin\theta)$$
$$= (a, a\tan\theta)$$

$$r'(\theta) = (0, a\sec^2\theta)$$

$$\vec{F}(r(\theta)) = \left\langle \frac{-a\tan\theta}{a^2\sec^2\theta}, \frac{a}{a^2\sec^2\theta} \right\rangle$$

$$\vec{F}(r(\theta))\cdot r'(\theta) = 1$$

Hence $\displaystyle\int_L \vec{F}\cdot dr = \int_{\theta_1}^{\theta_2} 1\, d\theta = \theta_2 - \theta_1$

Method 3 (Fundamental theorem of line integrals)

Notice that $f(x,y) = \arctan\left(\frac{y}{x}\right)$ is such that

$$f_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{-y}{x^2} = \frac{-y}{x^2+y^2}$$

$$f_y = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}.$$

Hence $\nabla f = \vec{F}$ as long as $(x,y)\neq 0$. In particular this holds in a neighbourhood of the line segment. Hence by fundamental theorem,

$$\int_L \vec{F}\cdot dr = f(a,c) - f(a,b) = \arctan\left(\frac{c}{a}\right) - \arctan\left(\frac{b}{a}\right) = \angle OQ - \angle OP.$$

$\blacksquare$

(b) Suppose that $\mathcal{C}$ is parametrized by $\boldsymbol{r}(\theta) = (f(\theta)\cos\theta, f(\theta)\sin(\theta))$ for $\theta_1 \leq \theta \leq \theta_2$.
Show that $\boldsymbol{F}(\boldsymbol{r}(\theta))\cdot\boldsymbol{r}'(\theta)\,d\theta = d\theta$. Use this to show that $\displaystyle\int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \theta_2 - \theta_1$.

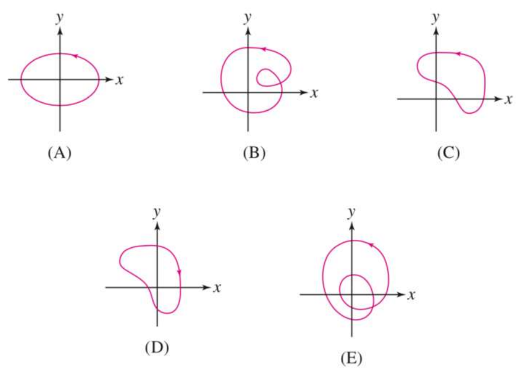
$r'(\theta) = f'(\theta)(\cos\theta, \sin\theta) + f(\theta)(-\sin\theta, \cos\theta)$

$$F(r(\theta)) = \left\langle -\frac{f(\theta)\sin\theta}{f^2(\theta)}, \frac{f(\theta)\cos\theta}{f^2(\theta)} \right\rangle$$

$F(r(\theta)) \cdot r'(\theta) = -\cos\theta\sin\theta + \sin\theta\cos\theta + \sin^2\theta + \cos^2\theta = 1.$

Hence $\int_C \vec{F} \cdot d\vec{r} = \int_{\theta_1}^{\theta_2} 1\, d\theta = \theta_2 - \theta_1$.

(c) Part (b) implies that if $\mathcal{C}$ is a closed path that winds around the origin n times (where n is negative if the path is in the clockwise direction) then $\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r} = 2\pi n$. The number n is called the *winding number* of the path, and it is important in various mathematical fields, especially in topology. Determine $\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r}$ for each of the curves in the pictures below.



part (b) doesn't actually imply this? well whatever.

a) 2π b) 2π c) 0 d) -2π e) 4π

(2) Consider the vector field $\boldsymbol{F} = \langle (1+xy)e^{xy}, x^2e^{xy} \rangle$.

(a) Compute $\operatorname{curl} \boldsymbol{G}$, where $\boldsymbol{G} = \langle (1+xy)e^{xy}, x^2e^{xy}, 0 \rangle$. Is $\boldsymbol{F}$ a conservative vector field?

(b) Depending on your answer to (a), either find a potential function for $\boldsymbol{F}$ or state that none exists.

(c) Evaluate $\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r}$ for

$$\mathcal{C} : \boldsymbol{r}(t) = (\cos t, 2\sin t), \qquad 0 \le t \le \pi/2, \qquad \text{(counterclockwise)}.$$

d) $$\nabla \times \vec{G} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ (1+xy)e^{xy} & x^2e^{xy} & 0 \end{vmatrix} = \langle 0, 0, 2xe^{xy} + yx^2e^{xy} - xe^{xy} - (1+xy)xe^{xy} \rangle$$

$$\begin{vmatrix} \partial_x & \partial_y & \partial_z \\ (1+xy)e^{xy} & x^2e^{xy} & 0 \end{vmatrix} = \langle 0, 0, \text{[illegible]} \rangle$$

$$= \langle 0, 0, 0 \rangle$$

Since G is defined for all real numbers (simply connected), it is conservative. Hence so is $\vec{F}$.

b). $\vec{F}(x,y) = \langle (1+xy)e^{xy}, x^2e^{xy} \rangle$

so if ϕ potential function, $\phi_x = (1+xy)e^{xy}$ $\cdots$ (1)

$\phi_y = x^2e^{xy}$ $\cdots$ (2)

From (2) $\phi = \int x^2 e^{xy}\,dy = xe^{xy} + C(x)$

so differentiating w.r.t x

$$\phi_x = e^{xy} + xye^{xy} + C'(x)$$

$$= (1+xy)e^{xy} + C'(x) = (1+xy)e^{xy} \text{ by (1)}.$$

Hence $C'(x) = 0$ so we can take $C = 0$ and hence

$\phi(x,y) = xe^{xy}$ is a potential function.

c) We have $r(0) = (1, 0)$, $r(\frac{\pi}{2}) = (0, 2)$.

Hence by fundamental theorem and above,

$$\int_C \vec{F} \cdot dr = \phi(0,2) - \phi(1,0) = 0 - 1 = -1.$$