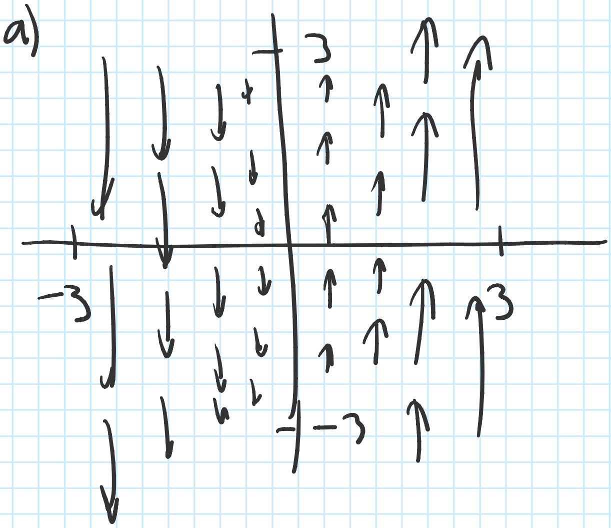
Worksheet 5

(1) Sketch the following planar vector fields by drawing the vectors attached to points with integer coordinates in the rectangle $-3 \le x \le 3,\ -3 \le y \le 3$.

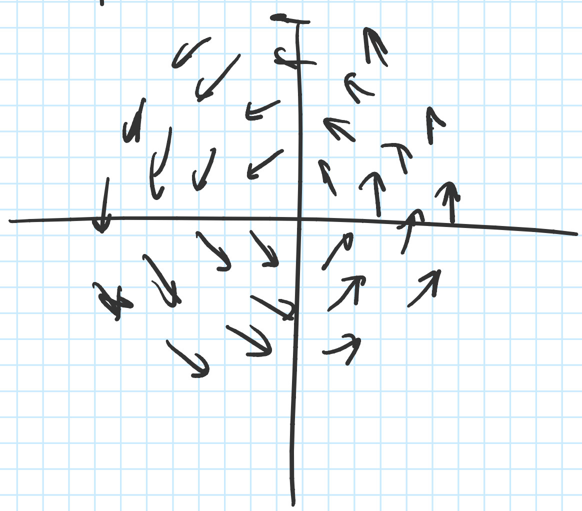
(a) $\boldsymbol{F} = \langle 0, x \rangle$

(b) $\boldsymbol{F} = \left\langle \dfrac{-y}{\sqrt{x^2+y^2}}, \dfrac{x}{\sqrt{x^2+y^2}} \right\rangle$

a)



b) unit vortex field



(2) Compute div$\boldsymbol{F}$ and curl $\boldsymbol{F}$ for $\boldsymbol{F} = \sin(x+z)\boldsymbol{i} - ye^{xz}\boldsymbol{k}$.

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(\sin(x+z)) + \frac{\partial}{\partial z}(-ye^{xz})$$

$$= \cos(x+z) - yxe^{xz}$$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ \sin(x+z) & 0 & -ye^{xz} \end{vmatrix} = \langle -e^{xz}, \cos(x+z) - yze^{xz}, 0 \rangle$$

(3) Show that any vector field of the form

$$\boldsymbol{F} = \langle f(x), g(y), h(z)\rangle$$

is conservative.

let $F(x)$, $G(y)$ and $H(z)$ be antiderivatives of f, g and h respectively.

Then $\nabla(F(x) + G(y) + H(z)) = \langle f(x), g(y), h(z)\rangle$ so conservative

(4) Suppose that $\boldsymbol{F}$ is a planar vector field that represents the force acting on an object at each point in the plane. Suppose an object is moving along a curve $\mathcal{C}$ in the plane. The total work performed by the field $\boldsymbol{F}$ is defined as the line integral

$$W = \int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{r},$$

where $\boldsymbol{r}$ is a parametrization of the curve $\mathcal{C}$.

(a) Suppose that the object is moving along the graph of $y = x^2$ from $(0,0)$ to $(1,1)$. Find a parametrization $\boldsymbol{r}(t)$ for this curve and compute $\boldsymbol{r}'(t)$.
(b) Suppose that the force field is given by $\boldsymbol{F} = \langle x+y, x-y\rangle$. Compute $\boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)$.
(c) Compute the work done by $\boldsymbol{F}$ along $\mathcal{C}$ by computing

$$\int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t)\, dt,$$

where $\boldsymbol{r}(a) = (0,0)$ and $\boldsymbol{r}(b) = (1,1)$.
(d) Do all of that again, assuming that the object moves along $x = y^2$ from $(0,0)$ to $(1,1)$.
(e) Is the vector field $\boldsymbol{F}$ conservative? (Later we will see that your answers to (c) and (d) are related to whether or not $\boldsymbol{F}$ is conservative).

a) $r(t) = \langle t, t^2\rangle$ for $t \in [0,1]$.

$r'(t) = \langle 1, 2t\rangle$.

b) $F(r(t)) \cdot r'(t) = \langle t+t^2, t-t^2\rangle \cdot \langle 1, 2t\rangle$

$= t + t^2 + 2t^2 - 2t^3$

$= t + 3t^2 - 2t^3$

c) $\int_0^1 F(r(t)) \cdot v'(t)\, dt = \int_0^1 t + 3t^2 - 2t^3 dt$

$= \frac{t^2}{2} + t^3 - \frac{1}{2}t^4 \Big|_0^1$

$= \frac{1}{2} + 1 - \frac{1}{2} = 1.$

$$= \frac{1}{2} + 1 - \frac{1}{2} = 1.$$

d) $r(t) = (t^2, t)$

$r'(t) = (2t, 1)$

$F(r(t)) = (t+t^2, t^2 - t)$

$$F(r(t)) \cdot r'(t) = 2t^2 + 2t^3 + t^2 - t$$
$$= -t + 3t^2 + 2t^3$$

$$\int_0^1 -t + 3t^2 + 2t^3 \, dt = -\frac{t^2}{2} + t^3 + \frac{1}{2} t^4 \Big|_0^1 = -\frac{1}{2} + 1 + \frac{1}{2} = 1$$

e) It is conservative. $\vec{F} = \langle x+y, x-y \rangle$. If ϕ is potential, then $\phi_x = x+y$, $\phi_y = x-y$.

$\phi = \int x+y \, dx = \frac{x^2}{2} + xy + c(y)$ and $\phi = \int x-y \, dy = xy - \frac{y^2}{2} + d(x)$.

Hence $\phi = \frac{x^2}{2} + xy - \frac{y^2}{2}$ works.

(5) Calculate $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$, where C is the circle of radius 2 centered at the origin (oriented counterclockwise), and

$$\boldsymbol{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle.$$

How does the answer change if the radius of the circle changes?

$r(t) = \langle 2\cos t, 2\sin t \rangle$

$r'(t) = \langle -2\sin t, 2\cos t \rangle = 2\langle -\sin t, \cos t \rangle$

$F(r(t)) = \left\langle -\frac{2\sin t}{4}, \frac{2\cos t}{4} \right\rangle = \frac{1}{2}\langle -\sin t, \cos t \rangle$

Hence $F(r(t)) \cdot r'(t) = 1$.

Therefore $\int_C \vec{F} \cdot dr = \int_0^{2\pi} 1 \, dt = 2\pi$.

Therefore $\int_C F \cdot dr = \int_0 1\, dt = 2\pi$.

This answer doesn't change with radius.

ie $r(t) = r(\cos t, \sin t)$

$r'(t) = r(-\sin t, \cos t)$

$F(r(t)) = \frac{1}{r}(-\sin t, \cos t)$

so $F(r(t)) \cdot r'(t) = 1$.