Worksheet 4

(1) Use the map
$$G(u,v) = \left(\frac{u+v}{2}, \frac{u-v}{2}\right)$$
to compute
$$\iint_{\mathcal{D}} ((x-y)\sin(x+y))^2 \, dxdy,$$
where $\mathcal{D}$ is the rectangle with vertices $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$, and $(0, \pi)$.

$G_u(u,v) = \left(\frac{1}{2}, \frac{1}{2}\right)$

$G_v(u,v) = \left(\frac{1}{2}, -1/2\right)$

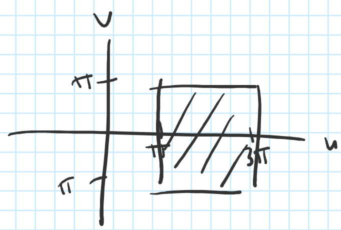
Hence $J(G) = \det\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1/2 \end{pmatrix} = -\frac{1}{2} \Rightarrow |J(G)| = \frac{1}{2}$.

Now we want to find $G^{-1}(D)$. We have $x = \frac{u+v}{2}$, $y = \frac{u-v}{2}$

so $u = x+y$, $v = x-y$ and so $G^{-1}(x,y) = (x+y, x-y)$. Moreover, this is linear so $G^{-1}(D)$ is a rectangle with vertices

$G^{-1}(\pi, 0) = (\pi, \pi)$, $G^{-1}(2\pi, \pi) = (3\pi, \pi)$, $G^{-1}(\pi, 2\pi) = (3\pi, \pi)$

$G^{-1}(0, \pi) = (\pi, \pi)$



Hence, after a change of variables:

$$\iint_D ((x-y)\sin(x+y))^2 \, dA = \int_{-\pi}^{\pi}\int_{\pi}^{3\pi} \left(\left(\frac{u+v}{2} - \frac{u-v}{2}\right)\sin\left(\frac{u+v}{2} + \frac{u-v}{2}\right)\right)^2 \frac{1}{2} \, du dv$$

$$= \int_{-\pi}^{\pi}\int_{\pi}^{3\pi} v^2 \sin(u)^2 \frac{1}{2} \, du dv$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} v^2 dv \int_{\pi}^{3\pi} \sin^2(u) \, du$$

$$= \frac{1}{2}\left(\frac{v^3}{3} \Big|_{-\pi}^{\pi} \right) \pi$$

$$= \frac{\pi}{2}\left(2\frac{\pi^3}{3}\right)$$

$$= \frac{\pi^4}{3}.$$

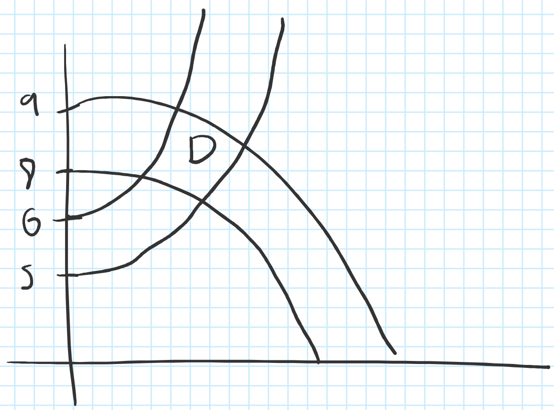
(2) Let $\mathcal{D}$ be the region in the first quadrant of the xy-plane bounded by the graphs of $y = 8 - x^2$, $y = 9 - x^2$, $y = x^3 + 5$, and $y = x^3 + 6$.

(a) Find a map F that maps $\mathcal{D}$ to a rectangle $\mathcal{R}$ in the uv-plane.

(b) Let G be the inverse of F (you do not need to compute G explicitly). Since $\mathcal{D}$ is small, we have

$$\text{area}(\mathcal{D}) \approx |\operatorname{Jac}(G)(u,v)|\,\text{area}(\mathcal{R}),$$

where (u, v) is any point in $\mathcal{R}$. (Note that if G were a linear transformation then $\approx$ would be $=$). Use this to approximate the area of $\mathcal{D}$ using the point $(x, y) = (1, 7)$.



a) The equations after rearranging are

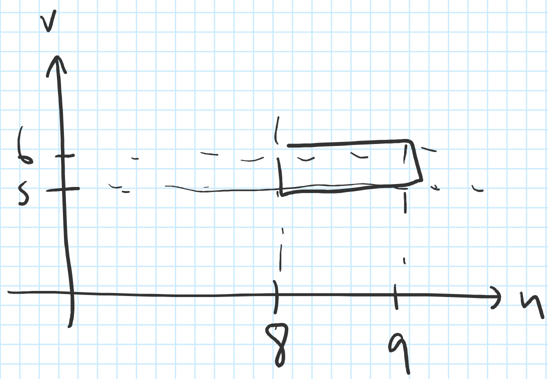
$y + x^2 = 8$, $y + x^2 = 9$

$y - x^3 = 5$ $y - x^3 = 6$

take $u = y + x^2$ $v = y - x^3$.

Then $F(x, y) = (y + x^2, y - x^3) = (u, v)$

maps D to rectangle R



b) $J(F) = \det\begin{pmatrix} 2x & 1 \\ -3x^2 & 1 \end{pmatrix} = 2x + 3x^2$

Then $J(G) = J(F)^{-1} = \dfrac{1}{2x + 3x^2}$

Hence $\text{Area}(D) \approx \dfrac{1}{5}$.

(3) Let $\mathcal{D}$ be the region in the first quadrant bounded by the curves $y = 2/x$, $y = 1/(2x)$, $y = 2x$, and $y = x/2$. Let F be the map $u = xy$, $v = y/x$ from the xy-plane to the uv-plane.

(a) Find the image of $\mathcal{D}$ under F.

(b) Let G be the inverse of F and compute the Jacobian of G.

(c) Show that

$$\iint_{\mathcal{D}} f\left(\frac{y}{x}\right) dxdy = \frac{3}{4}\int_{1/2}^{2} \frac{f(v)}{v}\, dv.$$

(d) Use this to evaluate

$$\iint_{\mathcal{D}} \frac{ye^{y/x}}{x}\, dxdy.$$

a) The curves rewritten are $xy = 2$, $xy = \frac{1}{2}$, $y/x = 2$, $\frac{y}{x} = 1/2$.

Hence it follows the image of these are $u = 1$, $u = 1/2$, $v = 2$, $v = 1/2$.

and so D is a rectangle

b) $J(F)(x,y) = \det\begin{pmatrix} y & x \\ -y/x^2 & 1/x \end{pmatrix} = \frac{y}{x} + \frac{y}{x} = 2\frac{y}{x}$.

so $J(G) = J(F)^{-1} = \frac{1}{2}\cdot\frac{x}{y} = \frac{1}{2v}$

c) $\iint_D f\left(\frac{y}{x}\right) dxdy = \int_{1/2}^{2}\int_{1/2}^{2} f(v)\cdot\frac{1}{2v}\, dudv$ note $\frac{1}{2v} > 0$.

$$= \frac{3}{4}\int_{1/2}^{2} \frac{f(v)}{v}\, dv$$

d) $\iint_D \frac{y}{x} e^{y/x}\, dxdy = \frac{3}{4}\int_{1/2}^{2} \frac{ve^{v}}{v}\, dv$ when $f(t) = te^{t}$.

$$= \frac{3}{4}\int_{1/2}^{2} e^{v}\, dv$$

$$= \frac{3}{4}\left(e^{2} - e^{1/2}\right)$$