Worksheet 2

(1) In this exercise, we will evaluate the following integral by reversing the order of integration.

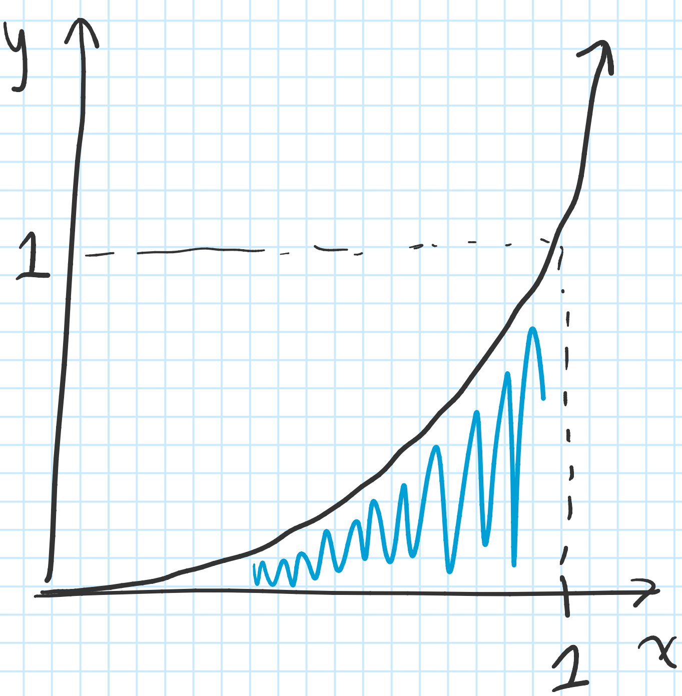
$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\, dx\, dy.$$

(a) Sketch the area represented by $\{(x,y) : 0 \le y \le 1, \sqrt{y} \le x \le 1\}$.

(b) Write a representation of this area in the form $\{(x,y) : a \le x \le b, f(x) \le y \le g(x)\}$, where a and b are numbers and f and g are functions of x.

(c) Evaluate the integral.

(a)



Note: $\sqrt{y} = x \Rightarrow y = x^2$

(b) For a fixed value of x, y-values between $0 \le y \le x^2$. all possible x-values are from 0 to 1.

Hence $R = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le x^2\}$.

c) $$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\, dx\, dy = \int_0^1 \int_0^{x^2} \sqrt{x^3+1}\, dy\, dx$$

$$= \int_0^1 x^2\sqrt{x^3+1}\,dx$$

let $u = x^3$, $du = 3x^2 dx$. when $x=0$, $u=0$

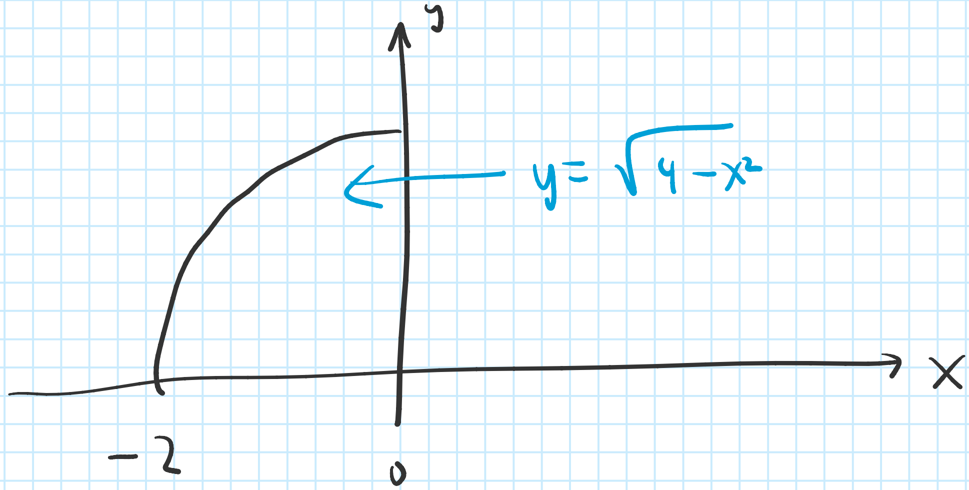
$x=1$, $u=1$. Hence integral becomes:

$$\frac{1}{3}\int_0^1 \sqrt{u+1}\,du = \frac{2}{9}(u+1)^{3/2}\Big|_0^1 = \frac{2}{9}\left(2^{3/2}-1\right)$$

(2) Use polar coordinates to evaluate

$$\int_{-2}^{0}\int_{0}^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx.$$

We are integrating over a quartercircle of radius 2



So polar coordinates:

$$\int^{0}\int^{\sqrt{4-x^2}} \cdots \qquad \int^{2}\int^{\pi} \cdots$$

$$\int_{-2}^{0}\int_{0}^{\sqrt{4-x^2}} x^2+y^2\,dy\,dx = \int_{0}^{2}\int_{\pi/2}^{\pi} r^2\cdot r\,d\theta\,dr$$

$$= \int_0^2 \frac{\pi}{2}\cdot r^3\,dr$$

$$= \frac{\pi}{8} r^4 \Big|_0^2$$

$$= 2\pi.$$

(3) Discuss with your group: what property of an *integrand* screams "polar coordinates!" to you? What property of a *region* screams "polar coordinates!"?

<u>integrand:</u> x^2+y^2 in integrand screams polar coordinates

<u>region:</u> circles, wedges. If the radius can be nicely expressed in terms of the angle.

(4) The function $P(x) = e^{-x^2}$ is fundamental in probability. It is called a Gaussian function or "bell curve." We will compute $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$ using the following brilliant strategy of Gauss: Instead of computing I, we will compute

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right).$$

(a) Convince yourself that

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dxdy = \lim_{R\to\infty}\iint_{D(R)} e^{-x^2-y^2}\,dxdy,$$

where $D(R)$ is the disk $x^2+y^2 \le R^2$.

(b) Convert the integral over $D(R)$ to polar coordinates.

(c) Evaluate to find I^2. Deduce the value of I.

(a) we are integrating the same function over the entirety of $\mathbb{R}^2$ and as $e^{-x^2-y^2} \to 0$ exponentially, it seems reasonable for them to be equal.

(b) $$\iint_{D(R)} e^{-x^2-y^2}\,dxdy = \int_0^R \int_0^{2\pi} e^{-r^2}\cdot r\,d\theta\,dr$$

$$= 2\pi \int_0^R r e^{-r^2}\,dr$$

$$= 2\pi \left. -\frac{1}{2} e^{-r^2} \right|_0^R$$

$$= -\pi e^{-R^2} + \pi$$

(c) Hence $$I^2 = \lim_{R\to\infty} \iint_{D(R)} e^{-x^2-y^2}\,dxdy$$

$$l \quad \pi \cdot {}^{-R^2} \quad \pi$$

$$= \lim_{R\to\infty} -\pi e^{-R^2} + \pi$$

$$= \pi .$$

Hence $I = \sqrt{\pi}$ (it must be +ve root since $e^{-x^2} > 0$).

(5) Let R be the region which lies above the x-axis and between the circles of radius 1 and 2 centered at $(0,0)$.

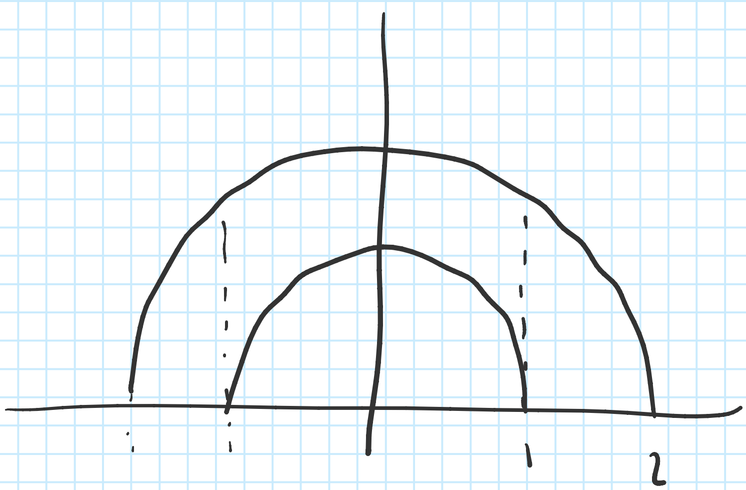
(a) Write the following integral as an sum of integrals in rectangular coordinates:

$$\iint_R y\, dA.$$

Do not evaluate these integrals.

(b) Evaluate the integral in part (a) using polar coordinates.

(a)



In rectangular coords we have to write this region piece-wise:

when $-2 \le x \le -1$, $0 \le y \le \sqrt{4-x^2}$

$-1 \le x \le 1$, $\sqrt{1-x^2} \le y \le \sqrt{4-x^2}$

$$1 \le x \le 2, \quad 0 \le y \le \sqrt{4-x^2}$$

Hence $$\iint_R y\,dA = \int_{-2}^{-1}\int_0^{\sqrt{4-x^2}} y\,dy\,dx + \int_{-1}^{1}\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} y\,dy\,dx + \int_1^2\int_0^{\sqrt{4-x^2}} y\,dy\,dx$$

b) $$\iint_R y\,dA = \int_0^{\pi}\int_1^2 r\sin\theta\, r\,dr\,d\theta$$

$$= \int_1^2 r^2\,dr \int_0^{\pi} \sin\theta\,d\theta$$

$$= \left(\frac{r^3}{3}\Big|_1^2\right)\left(-\cos\theta\Big|_0^{\pi}\right)$$

$$= \frac{14}{3}$$

(6) Compute $\displaystyle\int_0^\infty\int_0^\infty \frac{1}{(1+x^2+y^2)^2}\,dx\,dy.$

Let $S(R)$ be the quarter circle in first quadrant of radius R.

quadrant of radius R.

Then $$\int_0^\infty \int_0^\infty \frac{dx\,dy}{(1+x^2+y^2)^2} = \lim_{R\to\infty} \iint_{S(R)} \frac{dA}{(1+x^2+y^2)^2}$$

Now,

$$\iint_{S(R)} \frac{dA}{(1+x^2+y^2)^2} = \int_0^{\pi/2}\int_0^R \frac{r\,dr\,d\theta}{(1+r^2)^2}$$

$$= \frac{\pi}{2}\left(-\frac{1}{2}(1+r^2)^{-1}\Big|_0^R \right)$$

$$= \frac{\pi}{2}\left(-\frac{1}{2}(1+R^2)^{-1} + \frac{1}{2} \right)$$

Hence

$$\int_0^\infty \int_0^\infty \frac{dx\,dy}{(1+x^2+y^2)^2} = \lim_{R\to\infty} \frac{\pi}{2}\left(-\frac{1}{2}(1+R^2 + \frac{1}{2} \right)$$

$$= \frac{\pi}{4}.$$

$\blacksquare$