

Worksheet 10

(1) Let \mathbf{F} denote the *inverse square vector field*

$$\mathbf{F} = \frac{\langle x, y, z \rangle}{r^3},$$

where $r = \|\langle x, y, z \rangle\| = \sqrt{x^2 + y^2 + z^2}$. (Note that $\|\mathbf{F}\| = 1/r^2$.) The domain of \mathbf{F} is $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$

(a) Verify that $\frac{\partial F_1}{\partial x} = \frac{r^2 - 3x^2}{r^5}$. *Hint:* first show that $\frac{\partial r}{\partial x} = \frac{x}{r}$, then use the chain rule.

$$r = \sqrt{x^2 + y^2 + z^2}, \text{ so } \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}.$$

$$\begin{aligned} \text{Now, } \frac{\partial F_1}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{x}{r^3} \right) = \frac{\partial x}{\partial x} \cdot \frac{1}{r^3} + x \frac{\partial}{\partial x} \left(\frac{1}{r^3} \right) \text{ (product)} \\ &= \frac{1}{r^3} + -\frac{3x}{r^4} \cdot \frac{\partial r}{\partial x} \text{ (chain rule)} \\ &= \frac{1}{r^3} - \frac{3x^2}{r^5} \\ &= \frac{r^2 - 3x^2}{r^5} \text{ (Typo?)} \end{aligned}$$

(b) Show that $\operatorname{div}(\mathbf{F}) = 0$.

$$\text{By symmetry } \frac{\partial F_2}{\partial y} = \frac{r^2 - 3y^2}{r^5}, \quad \frac{\partial F_3}{\partial z} = \frac{r^2 - 3z^2}{r^5}$$

$$\text{Hence } \operatorname{div}(\vec{F}) = \frac{3r^2 - 3(x^2 + y^2 + z^2)}{r^5} = 0$$

- (c) Suppose that \mathcal{S} is a closed surface in \mathbb{R}^3 that does not enclose the origin. Show that the flux of \mathbf{F} through \mathcal{S} is zero. *Hint:* since the interior of \mathcal{S} does not contain the origin, it is in the domain of \mathbf{F} , so you can apply the divergence theorem.

By divergence theorem (let V be the interior of \mathcal{S})

$$\iint_{\mathcal{S}} \vec{F} \cdot d\vec{S} = \iiint_V \operatorname{div}(\vec{F}) dV = \iiint_V 0 dV = 0.$$

- (d) Now suppose that \mathcal{S} is a closed surface in \mathbb{R}^3 that encloses the origin. We will show that the flux of \mathbf{F} through \mathcal{S} equals 4π .

- (i) Let R be small enough so that the sphere \mathcal{S}_R of radius R is entirely contained in the interior of \mathcal{S} . If we parametrize \mathcal{S}_R using spherical coordinates, then the normal vector is $\mathbf{N} = R^2 \sin \phi \langle \cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi \rangle$. Show that $\mathbf{F} \cdot \mathbf{N} = \sin \phi$.

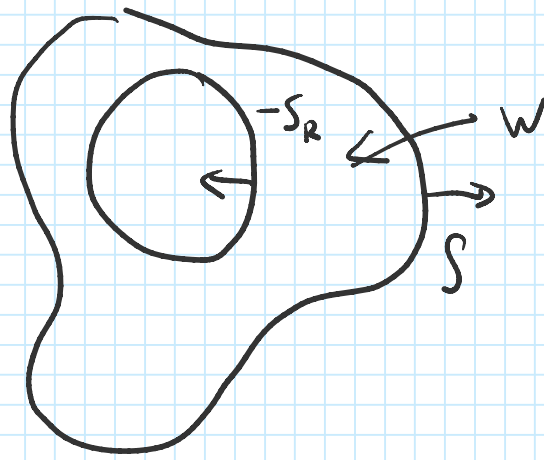
$$\vec{F}(G(\theta, \phi)) = \frac{1}{R^2} \langle \cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi \rangle.$$

$$\text{Hence } \vec{F}(G(\theta, \phi)) \cdot \vec{N} = \sin \phi \|\langle \cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi \rangle\|^2 \\ = \sin \phi$$

- (ii) Show that the flux of \mathbf{F} through \mathcal{S}_R equals 4π .

$$\begin{aligned} \iint_{\mathcal{S}_R} \vec{F} \cdot d\vec{S} &= \int_0^{2\pi} \int_0^\pi \sin \phi d\phi d\theta \\ &= 2\pi \int_0^\pi \sin \phi d\phi \\ &= 2\pi (-\cos \phi \Big|_0^\pi) \\ &= 4\pi \end{aligned}$$

- (iii) Let \mathcal{W} be the solid 3D region between \mathcal{S} and \mathcal{S}_R . Convince yourself that the boundary of \mathcal{W} is $\mathcal{S} - \mathcal{S}_R$ (i.e. \mathcal{S} is oriented with outward-pointing normals, while \mathcal{S}_R is oriented with inward-pointing normals).



It is $-S_R$ since above param has outward facing normal

- (iv) Show that the flux of \mathbf{F} through the boundary of \mathcal{W} is zero. *Hint:* since \mathcal{W} does not contain the origin, you can apply the divergence theorem.

Then last part (c).

- (v) Show that the flux of \mathbf{F} through \mathcal{S} equals 4π .

part (c) says we have

$$\iint_{S-S_R} \vec{F} \cdot d\vec{S} = 0 \Rightarrow \iint_S \vec{F} \cdot d\vec{S} = \iint_{S_R} \vec{F} \cdot d\vec{r} = 4\pi \text{ by (ii)}$$

- (e) Does \mathbf{F} have a vector potential \mathbf{A} defined on $\mathbb{R}^3 \setminus \{(0,0,0)\}$ such that $\mathbf{F} = \text{curl}(\mathbf{A})$?
Hint: suppose $\mathbf{F} = \text{curl}(\mathbf{A})$. What does Stokes' Theorem tell you about the flux of \mathbf{F} through any closed surface \mathcal{S} in \mathbb{R}^3 ?

If $\vec{F} = \text{curl}(\vec{A})$, then for any closed surface S we have

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \text{curl}(\vec{A}) \cdot d\vec{S}$$

$$= \int_{\partial \Omega} \vec{A} \cdot d\vec{s} \quad (\text{Stoke's})$$

$$= 0 \quad \text{since closed surface } (\partial \Omega = \emptyset)$$

But above we have $\iint_{S_R} \vec{F} \cdot d\vec{s} \neq 0$. Hence \vec{F} can't have a vector potential.

- (2) Let W be the region between the sphere of radius 4 and the cube of side length 1, both centered at the origin. What is the flux through $S = \partial W$ of a vector field \vec{F} whose divergence has the constant value -4 ?

$$\iint_{\partial W} \vec{F} \cdot d\vec{s} = \iiint_W \text{div}(\vec{F}) dV \quad (\text{divergence thm})$$

$$= \iiint_W -4 dV$$

$$= -4 \text{vol}(W)$$

Now, $\text{vol}(W) = \frac{4}{3} \pi 4^3 - 1$ and so we have some number.