Week 9 Notes

1. The equation of the **osculating** circle to a curve parametrized by $\vec{r}(t) = \langle f(t), g(t)\rangle$ at the **point** $(2, 1)$ is given by:

$$x^2 + y^2 - 4x + 2y + 1 = 0.$$

(Note that $(2, 1)$ is **not** the center of the osculating circle, it's the point where the osculating circle touches the curve $\vec{r}(t)$.)

(a) $\boxed{\text{5 points}}$ Find the curvature of the curve $\vec{r}(t)$ at the point $(2, 1)$.

(b) $\boxed{\text{3 points}}$ Find the unit normal $\vec{\mathbf{N}}$ to the curve $\vec{r}(t)$ at the point $(2, 1)$.

(c) $\boxed{\text{2 points}}$ Find a **vector parametrization** of the **tangent line** to $\vec{r}(t)$ at the point $(2, 1)$. (Note that this is **not** the same thing as finding the unit tangent vector $\vec{\mathbf{T}}$ at $(2, 1)$, you need to find a vector parametrization of the whole tangent line).

[**Hint:** First rewrite the equation as $(x-a)^2+(y-b)^2 = r^2$. Then look at the Figure 1 on Page 1 and use its geometry (i.e., the relation between the curve $\vec{r}(t)$ and its osculating circle) to solve this problem.]

a) $x^2+y^2-4x+2y+1=0$

$\Leftrightarrow (x-2)^2 - 4 + (y+1)^2 - 1 + 1 = 0$

$(x-2)^2 + (y+1)^2 = 4.$

Hence osc circle has radius $R=2$

and curvature $\kappa = 1/R = 1/2$

b) The circle has centre $(2,-1)$ and so by geometry,

$\vec{N} = (2,-1)-(2,1) / \|(2,-1)-(2,1)\|$

$= \dfrac{(0,-2)}{\|(0,-2)\|} = (0,-1)$

c) we need to find a vector perpendicular to $\vec{N}$. This will be the direction vector of the line.

$\vec{v} = \langle a,b \rangle$. $\vec{v}\cdot\vec{N} = 0$

$\Rightarrow -b = 0$ so $b=0$

$v = \langle a, b\rangle$. $v \cdot N = 0$

$\Rightarrow -b = 0$ so $b = 0$

and we can take $a = 1$.

Hence $\vec{v} = \langle 1, 0\rangle$ is the direction vector to the line.

$r(t) = (2,1) + t(1,0)$. is a parameterisation of tangent line.

2. $\boxed{\text{10 points}}$ Decompose the acceleration vector $\vec{\mathbf{a}}(t)$ of $\vec{\mathbf{r}}(t) = \langle t^2, 2t, \ln t\rangle$ into its tangential and normal components at $t = \frac{1}{2}$, i.e., find $a_{\vec{\mathbf{T}}(\frac{1}{2})}$, $\vec{\mathbf{T}}(\frac{1}{2})$, $a_{\vec{\mathbf{N}}(\frac{1}{2})}$ and $\vec{\mathbf{N}}(\frac{1}{2})$, and express $\vec{\mathbf{a}}(\frac{1}{2})$ as

$$\vec{\mathbf{a}}\left(\frac{1}{2}\right) = a_{\vec{\mathbf{T}}(\frac{1}{2})}\vec{\mathbf{T}}\left(\frac{1}{2}\right) + a_{\vec{\mathbf{N}}(\frac{1}{2})}\vec{\mathbf{N}}\left(\frac{1}{2}\right).$$

From textbook:

THEOREM 1 Tangential and Normal Components of Acceleration In the decomposition $\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}$, we have

$$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = \frac{\mathbf{a}\cdot\mathbf{v}}{\|\mathbf{v}\|}, \qquad a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = \sqrt{\|\mathbf{a}\|^2 - |a_{\mathbf{T}}|^2} \qquad \boxed{2}$$

and

$$a_{\mathbf{T}}\mathbf{T} = \left(\frac{\mathbf{a}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}, \qquad a_{\mathbf{N}}\mathbf{N} = \mathbf{a} - a_{\mathbf{T}}\mathbf{T} = \mathbf{a} - \left(\frac{\mathbf{a}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} \qquad \boxed{3}$$

So we need to first find $\vec{v}$ and $\vec{a}$.

$r(t) = (t^2, 2t, \ln t)$

$r'(t) = (2t, 2, \frac{1}{t})$ $\vec{v} = \vec{r}'(1/2) = (1, 2, 2)$ $\|\vec{v}\| = 3$

$r''(t) = (2, 0, -\frac{1}{t^2})$ $\vec{a} = \vec{v}'(1/2) = (2, 0, -4)$. $\|\vec{a}\| = \sqrt{20}$

$$a_T = \frac{a\cdot v}{\|v\|} = \frac{(2,0,-4)\cdot(1,2,2)}{\|(1,2,2)\|} = \frac{2-8}{\sqrt{1+4+4}} = -\frac{6}{3} = -2$$

$$a_{\vec{N}} = \sqrt{\|a\|^2 - |a_T|^2} = \sqrt{20-4} = 4$$

$$a_T T = \frac{a\cdot v}{v\cdot v} v = \frac{(2,0,-4)\cdot(1,2,2)}{9}(1,2,2) = -\frac{2}{3}(1,2,2)$$

$$a_N N = a - a_T T = (2,0,-4) + \frac{2}{3}(1,2,2) = \left(\frac{8}{3}, \frac{4}{3}, -\frac{8}{3}\right)$$

3. Let $\vec{\mathbf{r}}(t)$ be a parametrization of a curve, and $\vec{\mathbf{T}}(t)$, $\vec{\mathbf{N}}(t)$ and $\vec{\mathbf{B}}(t)$ are respectively the unit tangent, unit normal and the binormal vector.

   (a) $\boxed{\text{6 points}}$ Prove that $\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)$ is a **multiple** of $\vec{\mathbf{B}}(t)$.

   (b) $\boxed{\text{4 points}}$ Using Part (a) conclude that $\vec{\mathbf{B}}(t) = \dfrac{\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)}{||\vec{\mathbf{r}}'(t) \times \vec{\mathbf{r}}''(t)||}$.

   [**Hint:** You may use the fact that $\vec{\mathbf{r}}'(t) = v(t)\vec{\mathbf{T}}(t)$.]

a) $\vec{r}'(t) = v(t)\vec{T}(t)$

$\vec{r}''(t) = v'(t)\vec{T}(t) + v(t)\vec{T}'(t)$

$= v'(t)\vec{T}(t) + v(t)||\vec{T}'(t)||\vec{N}(t)$ since $\vec{N}(t) = \dfrac{\vec{T}'(t)}{||T'(t)||}$ by definition.

$\vec{r}'(t) \times r''(t) = \left(v(t)\vec{T}(t)\right) \times \left(v'(t)\vec{T}(t) + v(t)||\vec{T}'(t)||\vec{N}(t)\right)$

$= vv'\vec{T}\times\vec{T} + v^2||\vec{T}'(t)||\,\vec{T}\times\vec{N}$

$= v^2||\vec{T}'(t)||\,\vec{B}$ since $\vec{T}\times\vec{T} = 0$ and $\vec{B} = \vec{T}\times\vec{N}$ by defn

Hence $\vec{B}$ is a multiple of $r'\times r''$.

b). $\vec{B}$ has unit length, so if we divide $r'\times r''$ by $||r'\times r''||$ we have $\dfrac{r'\times r''}{||r'\times r''||} = \pm\vec{B}$. However, by above, $v^2||\vec{T}'|| > 0$ and so $\vec{B}$ is a positive multiple. (in same direction).

Hence $\vec{B} = \dfrac{r'\times r''}{||r'\times r''||}$.

Partial derivatives (possibly skipping ahead)

Given a function $f(x,y)$. The partial derivatives are the derivatives of each variable seperately. ie

$$\frac{\partial f}{\partial x} = f_x := \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{\partial f}{\partial y} = f_y = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

In practice, you essentially treat the other variable as constant and differentiate.

Example: $f(x,y) = x^2 y^5$.

$$\frac{\partial (x^2 y^5)}{\partial x} = y^5 \frac{\partial x^2}{\partial x} = y^5 (2x) = 2xy^5 \quad \text{(treat } y \text{ constant)}$$

$$\frac{\partial (x^2 y^5)}{\partial y} = x^2 \frac{\partial (y^5)}{\partial y} = x^2 (5y^4) = 5x^2 y^4 \quad \text{(treat } x \text{ constant)}$$

Question: Find the partials of the following:

a) $f(x,y) = \sin(x^2 y^3)$ b) $f(x,y) = x^3 + ye^x$

Answers:

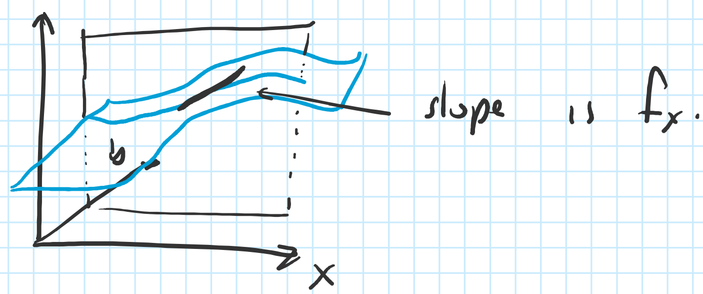
a) $f_x = \cos(x^2 y^3) \cdot (2xy^3) = 2xy^3 \cos(x^2 y^3)$

$f_y = 3x^2y^2\cos(x^2y^3)$

b) $f_x = 3x^2 + ye^x$

$f_y = e^x$

Geometrically, you can think of f_x as the rate of change or slope in the x direction.



similarly for f_y.

Textbook Q's: 8, 12