Week 8 Notes

Question 1. The involute of a circle has parameterisation given by

$$\vec{r}(\theta) = \langle R(\cos(\theta) + \theta \sin(\theta), R(\sin(\theta) - \theta \cos(\theta))\rangle$$

Find the arclength parameterisation.

$$r(\theta) = \langle R\cos\theta + R\theta\sin\theta,\ R\sin\theta - R\theta\cos\theta \rangle$$

$$r'(\theta) = \langle -R\sin\theta + R\sin\theta + R\theta\cos\theta,\ R\cos\theta - R\cos\theta + R\theta\sin\theta \rangle$$

$$= R\theta \langle \cos\theta, \sin\theta \rangle$$

Hence $s(\theta) = \int_0^\theta \| r'(u) \| \, du$

$$= \int_0^\theta Ru \, du$$

$$= R \frac{u^2}{2} \Big|_0^\theta$$

$$= \frac{R\theta^2}{2}$$

Hence, $\theta^{-1}(s) = \sqrt{\frac{2s}{R}}$ and the arclength parameterisation is given by

$$r_1(s) = r\left(\sqrt{\frac{2s}{R}}\right).$$

Question 2. Show that the curvature at an inflection point of a plane curve $y = f(x)$ is zero.

an inflection point is when $f''(x) = 0$. Hence using the formula for curvature of a plane curve:

$$\kappa(x) = \frac{|f''(x)|}{\quad}$$

$$\kappa(x) = \frac{|f''(x)|}{(1+f'(x)^2)^{3/2}}$$

we get $\kappa(x)=0$.

Question 3. Given a frenet frame $(\vec{T}, \vec{N}, \vec{B})$ with arclength paramareterisation.

(a) Show $\frac{d\vec{B}}{ds} = \vec{T} \times \frac{d\vec{N}}{ds}$ and conclude that $\frac{d\vec{B}}{ds}$ is orthogonal to $\vec{T}$.

(b) Show that $\frac{d\vec{B}}{ds}$ is orthogonal to $\vec{B}$. Hint: Differentiate $\vec{B} \cdot \vec{B} = 1$.

(c) Show that $\frac{d\vec{B}}{ds}$ is a multiple of $\vec{N}$.

a) By definition $\vec{B} = \vec{T} \times \vec{N}$, and so using product rule for cross products:

$$\frac{d\vec{B}}{ds} = \frac{d\vec{T}}{ds} \times \vec{N} + \vec{T} \times \frac{d\vec{N}}{ds} .$$

Since $\frac{d\vec{T}}{ds}$ is parallel to $\vec{N}$, $\frac{d\vec{T}}{ds} \times \vec{N} = 0$.

Hence $\frac{d\vec{B}}{ds} = \vec{T} \times \frac{d\vec{N}}{ds}$.

Since cross products output orthogonal vectors, we get $\frac{d\vec{B}}{ds} \perp \vec{T}$.

(b) $\vec{B} \cdot \vec{B} = 1$, differentiate via product rule gives

$$\frac{d\vec{B}}{ds} \cdot \vec{B} + \vec{B} \cdot \frac{d\vec{B}}{ds} = 0 \Rightarrow 2\vec{B} \cdot \frac{d\vec{B}}{ds} = 0.$$

Hence $\frac{d\vec{B}}{ds} \perp \vec{B}$.

c) Since $\frac{d\vec{B}}{ds} \perp \vec{B}$ and $\frac{d\vec{B}}{ds} \perp \vec{T}$, as $(\vec{T}, \vec{N}, \vec{B})$ form an ortho-normal system, $\frac{d\vec{B}}{ds}$ must be parallel to $\vec{N}$. ie a multiple.

Question 4. A particle has orbit given by

$$\vec{r}(t) = \langle \ln(t), t, t^2/2 \rangle \text{ for } t \geq 0.$$

Find the equation for the osculating plane to this particle at $t = 1$

we need to find a normal vector to the plane spanned by $\vec{T}, \vec{N}$. We can find this via $r' \times r''$ since r'' is in the osculating plane.

$$r'(t) = \langle \frac{1}{t}, 1, t \rangle$$

$$r''(t) = \langle -\frac{1}{t^2}, 0, 1 \rangle$$

$$r'(1) \times r''(1) = \langle 1, 1, 1 \rangle \times \langle -1, 0, 1 \rangle$$

$$= \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = \langle 1, -2, 1 \rangle.$$

Hence the osculating plane is given by (note $r(1) = \langle 1, 1, 1 \rangle$)

$$1 \cdot (x-1) - 2(y-1) + 1 \cdot (z-1) = 0$$

$$x - 2y + z = 0$$

Question 5. Show that for a vector function $\vec{r}(t)$, both $\vec{r}'(t)$ and $\vec{r}''$ lie in the osculating plane. Hint: differentiate $\vec{r}'(t) = v(t)\vec{T}(t)$.

$r'(t) = v(t)\vec{T}(t)$, Hence $r'(t)$ in osculating plane.

$$r''(t) = v'(t)\vec{T}(t) + \vec{T}'(t)$$

$$= v'(t)\vec{T}(t) + v(t)K(t)\vec{N}(t).$$

Hence $r''(t)$ in osculating plane.

Question 6. Find the domain for the following functions

(a) $f(x, y) = \dfrac{1}{\sqrt{x^2 + y^2 - 1}}$

(b) $f(x, y) = \dfrac{y \sin(x)}{1 + y}$

(c) $f(x, y) = -\dfrac{1}{\sin(xy)}$

(a) aslong as denominator is nonzero.

i.e. $\sqrt{x^2+y^2} = 1$ not included.

$x^2+y^2=1.$

So $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \neq 1\}$

(b). Similarly, $y \neq -1$.

(c) Similarly, $\sin(xy) \neq 0$

$xy \neq n\pi \quad n \in \mathbb{Z}.$

$D = \{(x,y) \in \mathbb{R}^2 \mid xy \neq n\pi, n \in \mathbb{Z}\}.$