Week7 Notes

Oscillating Circle:

- Given a curve $\vec{r}$. The oscillating circle at a pt is the circle that "best fits" the curve at that point.
- It fits best in the sense it's center is in the normal direction, it is tangent to the curve and has the same curvature.
- Since all circles have curvature $K_{circ} = \frac{1}{R}$ where R is the radius we can find the center Q of the oscillating circle at pt $r(t_0)$ by:

$$\vec{OQ} = \vec{r}(t_0) + \frac{1}{K(t_0)}\vec{N}.$$

Example: Find a parameterisation for the oscillating circle for $y = x^2$ at $x = 1/2$.

Step 1: Find the radius.

First, we parameterise by $r(t) = (t, t^2)$. Then calculate the curvature at $t = 1/2$.

$$K(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} = \frac{\|\langle 1, 2t, 0\rangle \times \langle 0, 2, 0\rangle\|}{(1+4t^2)^{3/2}} = \frac{2}{(1+4t^2)^{3/2}}.$$

Hence $K(1/2) = \frac{2}{2^{3/2}} = \frac{1}{\sqrt{2}}$. and so the radius of the oscillating circle is $R = \sqrt{2}$.

Step 2: Find $\vec{N}$ at $t = 1/2$.

$$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{\langle 1, 2t\rangle}{\sqrt{1+4t^2}}$$

$$T'(t) = -\frac{4t}{(1+4t^2)^{3/2}}\langle 1, 2t\rangle + \frac{1}{\sqrt{1+4t^2}}\langle 0, 2\rangle$$

$$= \frac{-4t\langle 1,2t\rangle + (1+4t^2)\langle 0,2\rangle}{(1+4t^2)^{3/2}}$$

$$= \frac{\langle -4t, 2\rangle}{(1+4t^2)^{3/2}}.$$

Hence $T'(1/2) = \frac{\langle -2, 2\rangle}{2\sqrt{2}} = \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle.$

and so $\|T'(1/2)\| = \sqrt{\frac{1}{2}+\frac{1}{2}} = 1$

Hence $\vec{N}(\frac{1}{2}) = \frac{1}{\sqrt{2}}\langle -1, 1\rangle.$

Step 3: Find center Q.

$$\vec{OQ} = \vec{r}(\tfrac{1}{2}) + R\vec{N}$$

$$= \langle \tfrac{1}{2}, \tfrac{1}{4}\rangle + \sqrt{2}\cdot\tfrac{1}{\sqrt{2}}\langle -1, 1\rangle$$

$$= \langle -\tfrac{1}{2}, 5/4\rangle$$

Step 4: Parameterize the circle.

we have the center $\langle -1/2, 5/4\rangle$ and radius $\sqrt{2}$.

Hence $r(t) = \langle -\frac{1}{2}, \frac{5}{4}\rangle + \sqrt{2}\langle \cos t, \sin t\rangle.$ ☒

Motion in 3-space.

we have: position $\vec{r}(t)$

velocity $\vec{r}'(t)$ or $\vec{v}(t)$

acceleration $\vec{r}''(t)$. or $\vec{a}(t)$

Observe that we can write velocity $\vec{v}(t) = v(t)\vec{T}(t)$ where $v(t) = \|\vec{v}(t)\|$ is the speed.

so $\vec{a}(t) = (\vec{v}(t))' = v'(t)\vec{T}(t) + v(t)\vec{T}'(t)$ by product rule.

Hence $\vec{a}(t)$ is also in the same plane as the tangent $\vec{T}$ and normal $\vec{N}$ and so we can decompose into tangent and normal components

$$\vec{a}(t) = a_T(t)\vec{T}(t) + a_N(t)\vec{N}(t).$$

THEOREM 1 Tangential and Normal Components of Acceleration In the decomposition $\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}$, we have

$$\boxed{a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = \frac{\mathbf{a}\cdot\mathbf{v}}{\|\mathbf{v}\|}, \qquad a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = \sqrt{\|\mathbf{a}\|^2 - |a_{\mathbf{T}}|^2}} \qquad \boxed{2}$$

and

$$\boxed{a_{\mathbf{T}}\mathbf{T} = \left(\frac{\mathbf{a}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}, \qquad a_{\mathbf{N}}\mathbf{N} = \mathbf{a} - a_{\mathbf{T}}\mathbf{T} = \mathbf{a} - \left(\frac{\mathbf{a}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}} \qquad \boxed{3}$$

Example

Find the tangential and normal component of $r(t) = \langle \frac{1}{3}t^3, 1-3t \rangle$ at $t=-1$.

<u>Answer</u>: we have $r'(t) = \langle t^2, -3 \rangle$

$$r''(t) = \langle 2t, 0 \rangle.$$

Hence $\vec{v} = r'(-1) = \langle 1, -3 \rangle$, $\vec{a} = r''(-1) = \langle -2, 0 \rangle$.

Therefore, $a_T = \dfrac{\vec{a}\cdot\vec{v}}{\|\vec{v}\|} = \dfrac{\langle -2,0 \rangle \cdot \langle 1,-3 \rangle}{\sqrt{10}} = \dfrac{-2}{\sqrt{10}}$

$$a_N = \sqrt{\|\vec{a}\|^2 - |a_T|^2} = \sqrt{4 - \frac{4}{10}} = \sqrt{\frac{36}{10}} = \frac{6}{\sqrt{10}}.$$

Hence $\vec{a} = -\dfrac{2}{\sqrt{10}}\vec{T} + \dfrac{6}{\sqrt{10}}\vec{N}$.

Question:

Given $r(t) = \langle t^2, 2t, \ln t \rangle$, decompose into tangential and normal components at $t = 1/2$

Answer: $r'(t) = \langle 2t, 2, 1/t \rangle$

$r''(t) = \langle 2, 0, -1/t^2 \rangle$.

Hence $\vec{v} = r'(1/2) = \langle 1, 2, 2 \rangle$ $\vec{a} = r''(\tfrac{1}{2}) = \langle 2, 0, -4 \rangle$.

so $a_T = \dfrac{\vec{a} \cdot \vec{v}}{\|\vec{v}\|} = \dfrac{\langle 1,2,2 \rangle \cdot \langle 2,0,-4 \rangle}{3} = \dfrac{2-8}{3} = -2$

$a_N = \sqrt{\|\vec{a}\|^2 - \|a_T\|^2} = \sqrt{4 + 16 - 4} = 4$.

Hence $\vec{a} = -2\vec{T} + 4\vec{N}$.

connection between curvature and normal acceleration.

if $v(t)$ is the speed of $r(t)$, we can also show that it's true that $a_T(t) = v'(t)$ and $a_N(t) = \kappa(t) v^2(t)$

Question: A space shuttle is orbitting the earth at an altitude of 400km with constant speed 28000 km/hr. (Earth radius 6378km) what is the magnitude of the shuttle's acceleration.

Answer: Since constant speed, $a = a_N$. ie has no tangential accel.

by above $a_N = \kappa(t) v^2(t)$ Since circle $\kappa(t) = \dfrac{1}{6378+400} = \dfrac{1}{6778}$

Hence $a_N = \dfrac{28000^2}{6778}$ km/hr^2

6778

Or $\quad a_N = \dfrac{28000^2}{6778} \times \dfrac{1000}{60^4}\ m/s^2 \cong 8.9\ m/s^2.$