Week 6 Notes

Parameterisations:

- curves can be parameterised in more than one way. (actually infinite ways)
  For instance, the circle S^1 can be paramaratised by
  $r_1(t) = (\cos t, \sin t)$ or $r_2(t) = (\cos t^3, \sin t^3)$.
  Both go around the circle but do so at different speeds at different times.
- We say a parameterisation $r(t)$ is an <u>arclength parameterisation</u> if $\|r'(t)\| = 1$ for all t.
- Given a parameterisation $r(t)$, the arclength (from 0 to time t). is given by the function
  $$g(t) = \int_0^t \|r'(u)\| du$$
  we can then construct an arclength parameterisation by
  $$r_1(s) = r(g^{-1}(s)).$$

Example: $r(t) = \langle \cos 4t, \sin 4t, 3t \rangle$ we find an arclength param.

$$g(t) = \int_0^t \|r'(u)\| du$$
$$= \int_0^t \|\langle -4\sin 4u, 4\cos 4u, 3\rangle\| du$$
$$= \int_0^t \sqrt{16\sin^2 4u + 16\cos^2 4u + 9}\, du$$
$$= \int_0^t 5du = 5t \quad \text{so} \quad g^{-1}(s) = \frac{s}{5}$$

Hence arclength param $r_1(s) = r(g^{-1}(s)) = \langle \cos \frac{4}{5}s, \sin \frac{4}{5}s, \frac{3}{5}s \rangle$.

Question:

25. Let $\mathbf{r}(t) = \langle 3t+1, 4t-5, 2t \rangle$.

(a) Evaluate the arc length integral $s(t) = \int_0^t \|\mathbf{r}'(u)\|\, du$.

(b) Find the inverse $g(s)$ of $s(t)$.

(c) Verify that $\mathbf{r}_1(s) = \mathbf{r}(g(s))$ is an arc length parametrization.

(a) $s(t) = \int_0^t \|\langle 3, 4, 2 \rangle\| \, du$

$= \int_0^t \sqrt{9+16+4}\, du$

$= \sqrt{29}\, t \Rightarrow t = \frac{s}{\sqrt{29}}$.

(b) Hence $g(s) = \frac{s}{\sqrt{29}}$

(c) $r_1(s) = r\left(\frac{s}{\sqrt{29}}\right) = \left\langle \frac{3}{\sqrt{29}} s + 1, \frac{4}{\sqrt{29}} s - 5, \frac{2}{\sqrt{29}} s \right\rangle$

and $r_1'(s) = \frac{1}{\sqrt{29}} \langle 3, 4, 2 \rangle$

$\|r_1'(s)\| = 1$. Hence arclength param

Curvature:

The curvature $K(t)$ is a measurement of how much a curve bends at a point. There are a few ways to calculate it but in practice, the easiest way is given by formula.

$$K(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$$ when $r(t)$ is some param.

Example: (or Q?) $r(t) = \langle r\cos t, r\sin t, 0 \rangle$ circle of radius r.

Then $r'(t) = \langle -r\sin t, r\cos t, 0 \rangle$

$$r''(t) = \langle -r\cos t, -r\sin t, 0\rangle$$

$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ -r\sin t & r\cos t & 0 \\ -r\cos t & -r\sin t & 0 \end{vmatrix} = \langle 0, 0, r^2\rangle$$

so $$K(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} = \frac{r^2}{r^3} = \frac{1}{r}.$$

<u>Question:</u> Given a graph $y = f(x)$. This is a curve in the xy-plane which we can consider as the xy-plane in 3-space. What is the curvature of a graph?

<u>Answer:</u> We can parameterize a graph by $x = t$, $y = f(t)$, $z = 0$.
ie $r(t) = \langle t, f(t), 0\rangle$.

Hence, $$r'(t) = \langle 1, f'(t), 0\rangle$$
$$r''(t) = \langle 0, f''(t), 0\rangle$$

$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ 1 & f'(t) & 0 \\ 0 & f''(t) & 0 \end{vmatrix} = f''(t)$$

Hence, $$K(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} = \frac{|f''(t)|}{(1 + f'(t)^2)^{3/2}}$$

<u>Question:</u> Find the curvature of the Cornu Spiral
$$x(t) = \int_0^t \sin\frac{u^2}{2}\,du \qquad y(t) = \int_0^t \cos\left(\frac{u^2}{2}\right)du.$$

<u>Answer:</u> $$r(t) = \left\langle \int_0^t \sin\frac{u^2}{2}\,du, \int_0^t \cos\frac{u^2}{2}\,du, 0\right\rangle$$

$$r'(t) = \langle \sin\frac{t^2}{2}, \cos\frac{t^2}{2}, 0 \rangle$$

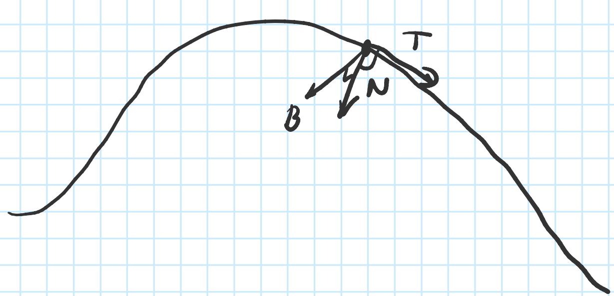
$$r''(t) = \langle t\cos\frac{t^2}{2}, -t\sin\frac{t^2}{2}, 0 \rangle$$

$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ \sin t^2/2 & \cos t^2/2 & 0 \\ t\cos t^2/2 & -t\sin t^2/2 & 0 \end{vmatrix} = -t$$

Hence, $k(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} = |t|$.

- Given a parameterisation $r(t)$, we have
  - unit tangent vector $T(t) = \frac{r'(t)}{\|r(t)\|}$
  - unit normal vector $N(t) = \frac{T'(t)}{\|T'(t)\|}$
  - The binormal vector $B(t) = T(t) \times N(t)$.

These form a right handed system $\{T, N, B\}$.



Question: Find T, N, B for $r(t) = \langle t, t, e^t \rangle$ at $(0,0,1)$

Answer: $r'(t) = \langle 1, 1, e^t \rangle$. $\|r'(t)\| = \sqrt{2 + e^{2t}}$

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Hence $T(t) = \frac{1}{\sqrt{2+e^{2t}}} \langle 1, 1, e^t \rangle$

$$T'(t) = \frac{-1}{\cancel{2}(2+e^{2t})^{3/2}} \cdot \cancel{2}e^{2t} \langle 1, 1, e^t \rangle + \frac{1}{\sqrt{2+e^{2t}}} \langle 0, 0, e^t \rangle$$

$$T'(0) = \frac{-1}{3^{3/2}} \langle 1, 1, 1 \rangle + \frac{1}{\sqrt{3}} \langle 0, 0, 1 \rangle$$

$$= \frac{1}{3\sqrt{3}} \left(-\langle 1, 1, 1 \rangle + \langle 0, 0, 3 \rangle \right)$$

$$= \frac{1}{3\sqrt{3}} \langle -1, -1, 2 \rangle.$$

$$\|T'(0)\| = \frac{\sqrt{4+2}}{3\sqrt{3}} = \frac{\sqrt{2}}{3}$$

Hence $N(0) = \frac{T'(0)}{\|T'(0)\|} = \frac{1}{\sqrt{6}} \langle -1, -1, 2 \rangle$

and $B(0) = T(0) \times N(0) = \left(\frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle \right) \times \left(\frac{1}{\sqrt{6}} \langle -1, -1, 2 \rangle \right)$

$$= \frac{1}{3\sqrt{2}} \langle 1, 1, 1 \rangle \times \langle -1, -1, 2 \rangle.$$

$$\langle 1, 1, 1 \rangle \times \langle -1, -1, 2 \rangle = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ -1 & -1 & 2 \end{vmatrix} = \langle 3, -3, 0 \rangle.$$

Hence, $B(0) = \frac{1}{3\sqrt{2}} \langle 3, -3, 0 \rangle$

Hence, $\{T, N, B\}$ at $t=0$ is $\left\{ \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle, \frac{1}{\sqrt{6}} \langle -1, -1, 2 \rangle, \frac{1}{3\sqrt{2}} \langle 3, -3, 0 \rangle \right\}$.

Question:

56. (a) What does it mean for a space curve to have a constant unit tangent vector **T**?

(b) What does it mean for a space curve to have a constant normal vector **N**?

(c) What does it mean for a space curve to have a constant binormal vector **B**?

(a) If T is constant, it must be moving in a line.

(b) Since $\|T\|=1 \Rightarrow T'$ is perpendicular to T. Moreover, N is alway pointing towards the centre of the oscillating circle so it follows that T must also be constant, since if this changes, it must do so in the plane perpendicular ~~of~~ to N but then the oscillating circle would be in this plane.

c) The curve must stay in the plane perpendicular to B and the bending of the curve must be on the same side as the direction always.