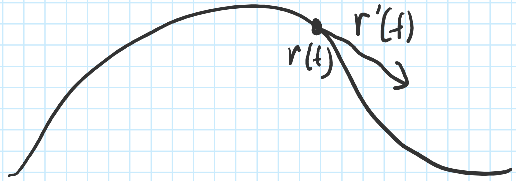
Week 5 Notes

Derivatives of vector valued functions

given $r(t) = \langle x(t), y(t), z(t) \rangle$, then $r'(t) = \langle x'(t), y'(t), z'(t) \rangle$.
Geometrically we think of $r'(t)$ as the tangent vector to the curve $r(t)$ at time t.



The derivative behaves similarly to the one-dimensional case. Given scalar function $f: \mathbb{R} \to \mathbb{R}$ we have:

product rule: $(f(t)r(t))' = f'(t)r(t) + f(t)r'(t)$

chain rule: $(r(f(t)))' = r'(f(t))r'(t)$

The dot and cross product also obey the product rule:

$$(r_1(t) \cdot r_2(t))' = r_1'(t) \cdot r_2(t) + r_1(t) \cdot r_2'(t)$$
$$(r_1(t) \times r_2(t))' = r_1'(t) \times r_2(t) + r_1(t) \times r_2'(t).$$

Group Question:

1) given $r_1(t) = \langle t^2, 1, 2t \rangle$ and $r_2(t) = \langle 1, 2, e^t \rangle$ calculate $(r_1(t) \cdot r_2(t))'$ in two ways:
   a) take dot product first and then differentiate
   b) differentiate using product rule

2) If $\|r(t)\|$ is constant, show that $r(t)$ and $r'(t)$ are orthogonal.
   Hint: $\|r(t)\|^2 = r(t) \cdot r(t)$

3) Show that $(\underline{a} \times r(t))' = \underline{a} \times r'(t)$ for any constant vector $\underline{a}$.

Answers:

1) a) $r_1(t) \cdot r_2(t) = t^2 + 2 + 2te^t$

so $(r_1(t) \cdot r_2(t))' = 2t + 2e^t + 2te^t$

b) $(r_1(t) \cdot r_2(t))' = r_1'(t) \cdot r_2(t) + r_1(t) \cdot r_2'(t)$

$$= \langle 2t, 0, 2 \rangle \cdot \langle 1, 2, e^t \rangle + \langle t^2, 1, 2t \rangle \cdot \langle 0, 0, e^t \rangle$$

$$= 2t + 2e^t + 2te^t.$$

2) $K = \|r(t)\|^2$ since constant.

Hence, $K = r(t) \cdot r(t)$ and differentiating:

$$0 = r'(t) \cdot r(t) + r(t) \cdot r'(t)$$

$\Rightarrow r'(t) \cdot r(t) = 0$

$\Rightarrow r'(t)$ and $r(t)$ orthogonal.

3) note, we can think of $\underline{a}$ as a constant vector function $g(t) = \underline{a}$. ie $g'(t) = 0$. Hence by product rule,

$$(\underline{a} \times r(t))' = (g(t) \times r(t))'$$

$$= g'(t) \times r(t) + g(t) \times r'(t)$$

$$= \underline{a} \times r'(t) \quad \text{since } g'(t) = 0.$$

32amid1prac

Midterm 2 (Practice Test)
Calculus of Several Variables
(Math 32A)

Name: ______________________ U ID: ____________

Question:	1	2	3	4	5	Total
Points:	5	5	5	5	5	25
Score:						

1. $\boxed{\text{5 points}}$ Find the equation of the plane which contains the point $(-1, 0, 2)$ and is parallel to the plane $2x - y - z = 3$.

parrallel to plane means same normal vector.
ie $\vec{n} = \langle 2, -1, -1 \rangle$.
Hence the plane is of the form $2x - y - z = d$ for some d. Since contains $(-1, 0, 2)$ we get

$$2(-1) - 0 - 2 = d$$

$$\therefore d = -4.$$

Therefore, the plane is: $2x - y - z = -4$.

2. $\boxed{\text{5 points}}$ Let $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ be two unit vectors such that $||\vec{\mathbf{u}} + \vec{\mathbf{v}}|| = \frac{3}{2}$. Then compute $||\vec{\mathbf{u}} - \vec{\mathbf{v}}||$.

use that as

$$||\vec{u}+\vec{v}||^2 = ||\vec{u}||^2 + 2\vec{u}\cdot\vec{v} + ||\vec{v}||^2$$

$$||\vec{u}-\vec{v}||^2 = ||\vec{u}||^2 - 2\vec{u}\cdot\vec{v} + ||\vec{v}||^2$$

Hence, $||\vec{u}+\vec{v}||^2 + ||\vec{u}-\vec{v}||^2 = 2||\vec{u}||^2 + 2||\vec{v}||^2$.

Now, we are given $||\vec{u}+\vec{v}||^2 = \frac{9}{4}$ and $||\vec{u}||^2 = ||\vec{v}||^2 = 1$.

Therefore, $\frac{9}{4} + ||\vec{u}-\vec{v}||^2 = 4$

$$||\vec{u}-\vec{v}||^2 = \frac{7}{4}$$

$$\therefore ||\vec{u}-\vec{v}|| = \frac{\sqrt{7}}{2}$$

3. $\boxed{\text{5 points}}$ Consider the following parametric equation:

$$x = a\cos\theta + a\sin\theta,\ y = -b\sin\theta + b\cos\theta, \quad \text{where } \theta \text{ is a parameter.}$$

Find a relation between x and y by eliminate the parameter θ.

We have that

$$\left(\frac{x}{a}\right)^2 = \cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta = 1 + 2\cos\theta\sin\theta$$

$$\left(\frac{y}{b}\right)^2 = \sin^2\theta - 2\cos\theta\sin\theta + \cos^2\theta = 1 - 2\cos\theta\sin\theta$$

Hence $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$

4. $\boxed{\text{5 points}}$ Find the solution of the differential equation with respect to the given initial conditions:

$$\vec{\mathbf{r}}''(t) = \langle e^t, \sin t, \cos t \rangle, \quad \vec{\mathbf{r}}(0) = \langle 1, 0, 1 \rangle \text{ and } \vec{\mathbf{r}}'(0) = \langle 0, 2, 2 \rangle.$$

$$r'(t) = \int r''(t)\,dt = \langle e^t, -\cos t, \sin t \rangle + \vec{c}$$

when $t=0$, $\quad r'(0) = \langle 0, 2, 2 \rangle = \langle 1, -1, 0 \rangle + \vec{c}$

$$\therefore \vec{c} = \langle -1, 3, 2 \rangle.$$

so $r'(t) = \langle e^t - 1, 3 - \cos t, 2 + \sin t \rangle$

Repeating,

$$r(t) = \int r'(t)\,dt = \langle e^t - t, 3t - \sin t, 2t - \cos t \rangle + \vec{c}$$

$$r(0) = \langle 1, 0, 1 \rangle = \langle 1, 0, -1 \rangle + \vec{c}$$

Hence $\quad \vec{c} = \langle 0, 0, 2 \rangle$

$$\therefore r(t) = \langle e^t - t, 3t - \sin t, 2 + 2t - \cos t \rangle$$

5. $\boxed{\text{5 points}}$ If $\vec{\mathbf{r}}(t)$ is a vector of constant length for all t, then prove that $\vec{\mathbf{r}}(t)$ is orthogonal to $\vec{\mathbf{r}}'(t)$.

As above