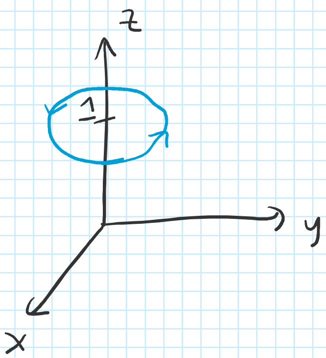
Vector-Valued Functions.

- A function of the form $r(t) = \langle x(t), y(t), z(t) \rangle = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$
- Think of t as time, $r(t)$ as a moving positional vector that traces out some path in $\mathbb{R}^3$. ie, vector-valued functions are really parameterisations of some space curve.

Example: $r(t) = \langle \cos t, \sin t, 1 \rangle \quad -\infty < t < \infty$.



The first two components are just the parameterisation of the circle and z is fixed at 1. Hence the curve traced by $r(t)$ is a circle at height 1. The path followed by $r(t)$ is that at $t=0$ it starts at the point $(1,0,1)$ and as $t \to \infty$ it goes around counterclockwise.

Group Questions:

Given two paths $r_1(t)$, $r_2(t)$. We say they intersect if there is a point P lying on both curves. We say that $r_1(t)$, $r_2(t)$ collide if $r_1(t_0) = r_2(t_0)$ for some t_0.

Is it true that:

1) If $r_1(t)$, $r_2(t)$ intersect, then they collide?
2) If $r_1(t)$, $r_2(t)$ collide then they intersect?
3) Intersection depends only on the curves traced by r_1, r_2. While collision depends only on underlying parameterisations?

Given
$$r_1(t) = (t^2+3,\ t+1,\ 6t^{-1})$$
$$r_2(t) = (4t,\ 2t-2,\ t^2-7)$$

determine if these collide or intersect?

Answer:

1) false 2) true 3) true.

They collide if $r_1(t) = r_2(t)$ has a solution.

components $\Rightarrow$ $t^2 + 3 = 4t$ ... (1)

$t + 1 = 2t - 2$ ... (2)

$6t^{-1} = t^2 - 7$ ... (3)

So (2): $t+1 = 2t-2 \Rightarrow t = 3$, plug into (1): $12 = 12$

plug into (3): $2 = 2$

Hence $t=3$ is a solution and the two curves collide. Since they collide, they also intersect.

Parameterising intersection of surfaces.

Example (From textbook)

How do we parameterise the intersection of the two surfaces: $x^2 - y^2 = z - 1$ and $x^2 + y^2 = 4$.

First way: try and write two of the variables in terms of the last one and make this the parameterising variable.

In this example, we write y and z in terms of just x.

$x^2 + y^2 = 4 \Rightarrow y = \pm\sqrt{4 - x^2}$

so $x^2 - y^2 = z - 1 \Rightarrow z = 1 + x^2 - y^2$

$= 1 + x^2 - (4 - x^2)$

$= 2x^2 - 3$.

Hence, let $x = t$. We then parametrise the intersection by the two paths:

$r_1(t) = (t, \sqrt{4-t^2}, 2t^2 - 3)$, $r_2(t) = (t, -\sqrt{4-t^2}, 2t^2 - 3)$.

Second method: use a known parameterisation

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the equation $x^2+y^2=4$ can be parameterized by $x=2\cos t$, $y=2\sin t$.
Hence, subbing these into the second equation gives

$$z = 1+y^2-x^2 = 1+4\sin^2 t - 4\cos^2 t.$$

Hence we can parameterise the intersection as

$$r(t) = (2\cos t, 2\sin t, 1+4\sin^2 t - 4\cos^2 t)$$

Questions

parameterise the intersection of the two cylinders $x^2+y^2=1$ and $x^2+z^2=1$.

Answer:

$x^2+y^2=1$ can be parameterized by $x=\cos t$, $y=\sin t$ and then by the second equation $z^2=1-x^2 \Rightarrow z=\pm\sqrt{1-x^2}$. Hence we get the two paths $z=\pm\sqrt{1-\cos^2 t}=\pm\sqrt{\sin^2(t)}=\pm|\sin(t)|$.
Hence we get the two paths:

$$r_1(t)=(\cos t, \sin t, |\sin(t)|), \quad r_2(t)=(\cos t, \sin(t), -|\sin(t)|)$$

Question:

The intersection of the surfaces:

$$z=x^2-y^2 \qquad z=x^2+xy-1.$$

Answer: we have on the intersection $x^2-y^2 = x^2+xy-1$

$$1 = y^2+xy$$

$$\Rightarrow \frac{1}{y} = y+x \Rightarrow x=\frac{1}{y}-y. \quad \text{Then } z=\left(\frac{1}{y}-y\right)^2-y$$

$$= \frac{1}{y^2}-2+y^2-y^2 = \frac{1}{y^2}-2 \Rightarrow z=\frac{1}{y^2}-2.$$

Hence let $y=t$ and then the parameterisation is:

$$r(t) = \left(\frac{1}{t} - t, t, \frac{1}{t^2} - 2 \right)$$