Week 2 Notes

corrected office hours:

Tuesday 3-4pm MATH 32A

Wednesday 3-4pm MATH 32B.

Today: <u>Dot and cross product.</u>

Reminder: Dot product.

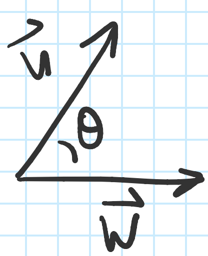
Takes two vectors and outputs a <u>scalar</u>

<u>algebraic</u>: $\vec{v} = \langle a, b, c \rangle$, $\vec{w} = \langle u, v, w \rangle$.

Then $\vec{v} \cdot \vec{w} = \langle a, b, c \rangle \cdot \langle u, v, w \rangle$

$$= au + bv + cw.$$

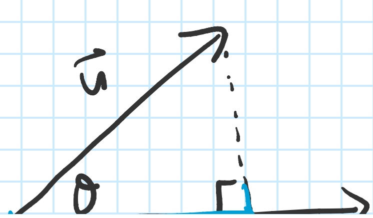
<u>Geometric</u>:



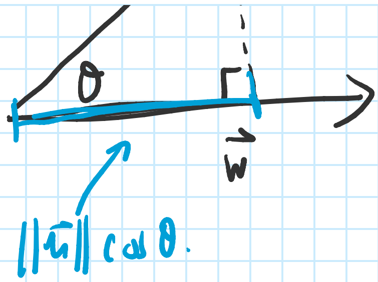
Then $\vec{u} \cdot \vec{w} = \|\vec{u}\| \|\vec{w}\| \cos\theta$.

what does this mean geometrically?

$\|\vec{u}\| \cos\theta$ is the component of u in the direction of $\vec{w}$



Hence $\vec{u} \cdot \vec{w}$ = component of $\vec{u}$ in the direction of $\vec{w}$ scaled by the length of $\vec{w}$.



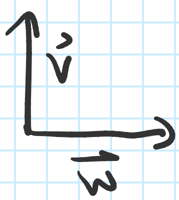
in the direction of w scaled
by the length of $\vec{w}$.

Question:

Draw vectors $\vec{v}$ and $\vec{w}$ such that
a) $\vec{v} \cdot \vec{w} = 0$, b) $\vec{v} \cdot \vec{w} < 0$ and c) $\vec{v} \cdot \vec{w} > 0$.

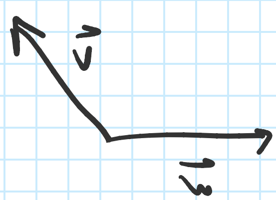
Answer:

a)



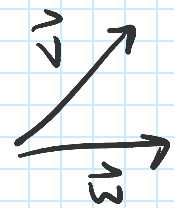
ie perpendicular.

b)



obtuse

c)



acute.

Extra properties:

For any vector $\vec{v}$, $||\vec{v}||^2 = \vec{v} \cdot \vec{v}$.

We can bring scalars out of dot products

ie $\lambda(\vec{v} \cdot \vec{w}) = (\lambda\vec{v}) \cdot \vec{w} = \vec{v} \cdot (\lambda\vec{w})$

We can also distribute like normal products

$(\vec{v} + \vec{u}) \cdot \vec{w} = \vec{v} \cdot \vec{w} + \vec{u} \cdot \vec{w}$ and

$\vec{u} \cdot (\vec{w} + \vec{v}) = \vec{u} \cdot \vec{w} + \vec{u} \cdot \vec{v}$.

We can use these algebraic properties to prove geometric facts.

Example

Given vectors $\vec{v}$ and $\vec{w}$, $\vec{v}$ orthogonal to $\vec{w}$ if and only if $\|\vec{v}-\vec{w}\| = \|\vec{v}+\vec{w}\|$

Proof:

$$\|\vec{v}+\vec{w}\|^2 = (\vec{v}+\vec{w})\cdot(\vec{v}+\vec{w})$$
$$= \vec{v}\cdot\vec{v} + \vec{w}\cdot\vec{v} + \vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{w}$$
$$= \|\vec{v}\|^2 + 2\vec{v}\cdot\vec{w} + \|\vec{w}\|^2$$

Similarly

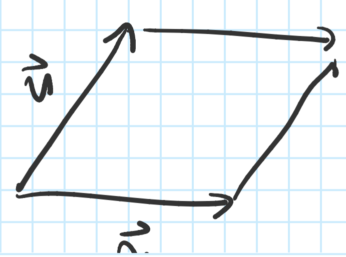
$$\|\vec{v}-\vec{w}\|^2 = \|\vec{v}\|^2 - 2\vec{v}\cdot\vec{w} + \|\vec{w}\|^2$$

Hence $\|\vec{v}+\vec{w}\|^2 - \|\vec{v}-\vec{w}\|^2 = 4\,\vec{v}\cdot\vec{w}$.

Hence, $\vec{v}\perp\vec{w} \iff \vec{v}\cdot\vec{w} \iff \|\vec{v}+\vec{w}\| = \|\vec{v}-\vec{w}\|$.

Question. Prove that the diagonals of a parallelogram are perpendicular if and only if the sides are equal.

Hint:



diagonals are $\vec{u}+\vec{v}$, $\vec{u}-\vec{v}$

so $(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) = ?$

$\vec{v}$

<u>Answer</u>: $(\vec{u}+\vec{v})\cdot(\vec{u}-\vec{v}) = \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{u} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{v}$

$$= \|\vec{u}\|^2 - \|\vec{v}\|^2$$

Hence, $\vec{u}+\vec{v} \perp \vec{u}-\vec{v} \iff \|\vec{u}\|^2 = \|\vec{v}\|^2$

$$\iff \|\vec{u}\| = \|\vec{v}\|.$$

ie the diagonals are $\perp \iff$ sides have same length ☺

<u>Cross Product:</u>

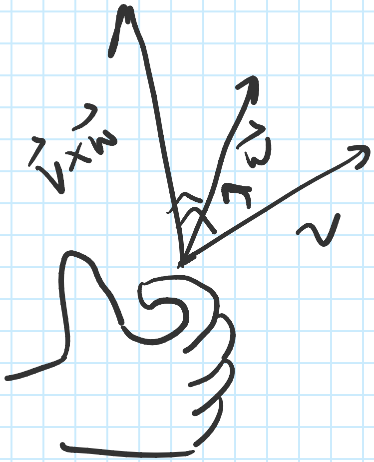
The cross product is very different to any product you have seen before

vector x vector $\longrightarrow$ vector.

unlike dot product, it outputs a <u>vector</u> not a scalar.

<u>Geometric description:</u>

$\vec{u} = \vec{v} \times \vec{w}$: This is a vector perpendicular to both $\vec{v}$ and $\vec{w}$, in the direction given by the right hand rule.



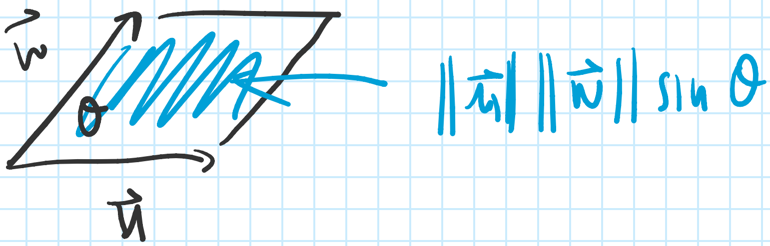
· Warnings:

the order matters!

$\vec{v} \times \vec{w} \neq \vec{w} \times \vec{v}$

· cross products only work in 3-dimensions.

The length of $\vec{u} \times \vec{w}$ is given by $\|\vec{u}\| \|\vec{w}\| \sin\theta$, (like dot, but sin instead) this is the area of the resulting parallelogram



Question:

When is $\vec{u} \times \vec{w} = 0$?

Answer: when $\vec{u} \parallel \vec{w}$.

Question: $\hat{i}, \hat{j}, \hat{k}$ are the unit vectors in the positive x, y and z-axis respectively

What is: a) $\hat{i} \times \hat{j} = ?$ b) $\hat{j} \times \hat{i} = ?$

c) $\hat{i} \times \hat{k} = ?$

Answer:

a) $\hat{k}$, b) $-\hat{k}$, c) $-\hat{j}$.

If there is time:

Many ways to calculate determinants: Here is one way!

Say we want to find $\begin{vmatrix} 3 & 2 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & -1 \end{vmatrix}$

repeat the the determinant for two more columns and add right down diagonals and subtract down left diagonals.

$$\left|\begin{matrix} 3 & 2 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & -1 \end{matrix}\right| \begin{matrix} 3 & 2 \\ 1 & 1 \\ 3 & 1 \end{matrix}$$

$$\begin{matrix} & & -3 & 0 & 1 \\ 3 & 0 & -2 & & \end{matrix}$$

so $\begin{vmatrix} 3 & 2 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & -1 \end{vmatrix} = -3 + 0 + 1 - 3 - 0 - (-2)$

$= -2 - 1 = -3$.

Question: cross product are calculated via determinants:

$$(2,1,0)\times(1,-1,1) := \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = ?$$

<u>Answer:</u>

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix}\begin{matrix} \hat{i} & \hat{j} \\ 2 & 1 \\ 1 & -1 \end{matrix}$$

$\hat{i}$ $\quad 0\hat{j}$ $\quad -\hat{k}$

$\hat{k}$ $\quad 0\hat{i}$ $\quad 2\hat{j}$

$$= \hat{i} - \hat{k} - \hat{k} - 2\hat{j}$$

$$= \hat{i} - 2\hat{j} - 2\hat{k}.$$

Hence, $(2,1,0)\times(1,-1,1) = \hat{i} - 2\hat{j} - 2\hat{k}$

$$= \langle 1, -2, -2 \rangle$$

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