

Learning Outcomes

- (1) Understand what directional derivatives mean geometrically
- (2) Know how to apply the limit definition for directional derivatives
- (3) Know what the gradient is and basic geometric properties.
- (4) Can use the gradient to calculate directional derivatives
- (5) (If time) understand how to use the gradient to find tangent planes.

Directional derivatives

- The partial derivatives $\partial_x f$, $\partial_y f$ are the rate of change (or slope) in the positive x or y-direction. ie, in the direction of the i or j vector.

- Directional derivatives are the same except in the direction given by a **unit** vector \vec{v} .

Example — go through Desmos Ap on directional derivatives.

Definition: For $\vec{v} = \langle h, k \rangle$ a **unit** vector, the directional derivative is defined as the limit:

$$D_{\vec{v}} f(a, b) = \lim_{t \rightarrow 0} \frac{f(a+th, b+tk) - f(a, b)}{t}$$

Example Find $D_{\vec{v}} f(0, 0)$ where $\vec{v} = \langle 2, 2 \rangle$.
and $f(x, y) = x^2 + 2y$

\vec{v} isn't a unit, so we first need to normalise it to get $\hat{\vec{v}} = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$

Then,

$$D_{\vec{v}} f(0, 0) = \lim_{t \rightarrow 0} \frac{f(0+t/\sqrt{2}, 0+t/\sqrt{2}) - f(0, 0)}{t}$$

$$= \lim_{t \rightarrow 0} \frac{(t/\sqrt{2})^2 + 2(t/\sqrt{2})^2}{t}$$

$$= \lim_{t \rightarrow 0} \frac{t^2/2 + t^2}{t} = \lim_{t \rightarrow 0} \frac{\frac{3}{2}t}{t} = 0.$$

Hence $D_{\vec{v}} f(0,0) = 0$.

Problem 1 Find the directional derivative

of $f(x,y) = xy - 4$ at $P=(2, -1)$ in the direction of $\vec{v} = \langle 1, -1 \rangle$.

Gradient

The gradient of a function is just the vector made from the partial derivatives.

- For 2-variable functions $f(x,y)$.

The gradient is $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$. These are vectors in the xy -plane.

- For 3-variable functions $f(x,y,z)$, the

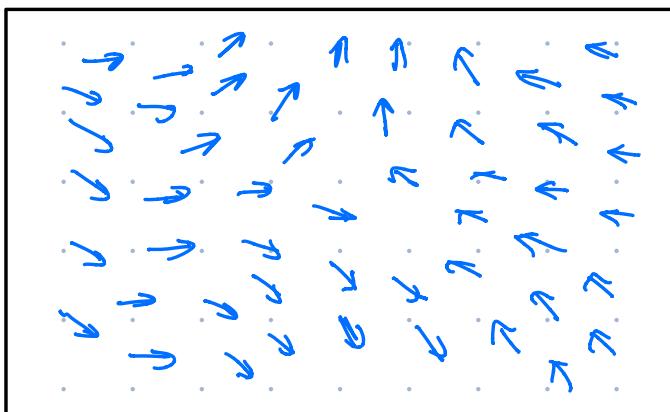
gradient is $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$. Vector

In 3-space.

- Similar for higher dimensions.

We can think of these as defining a "vector field". ie for $f(x,y)$, $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$ and at each point P in the plane, there is an attached vector given $\nabla f_P = \left\langle \frac{\partial f}{\partial x}(P), \frac{\partial f}{\partial y}(P) \right\rangle$

ie, looks like:



Example Go through gradient Desmos ap.

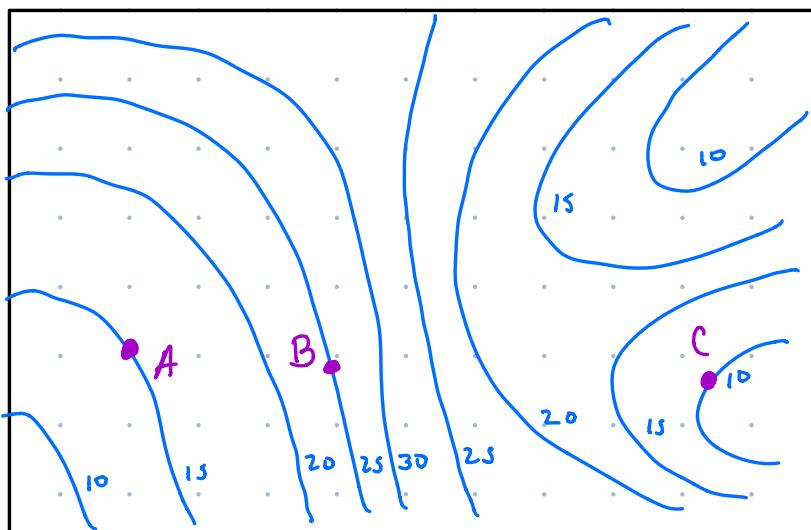
Gradients have a number of important properties:

- ∇f_P points in the direction of fastest increase.

- $-\nabla f_p$ " decrease.
- ∇f_p is always perpendicular to the level curves ie, perpendicular to the directions of no increase.

→ Look at these in the Desmos Ap →

Problem 2 Consider the following contour plot.
For each point, determine the direction of the gradient.



- We can use gradients to calculate directional derivatives.

If \vec{u} is a unit vector, the directional derivative at $P = \langle a, b \rangle$ in direction of \vec{u} is:

$$D_{\vec{u}} f(a, b) = \nabla f_p \cdot \vec{u}$$

which is also:

$$D_{\vec{u}} f(a, b) = \|\nabla f_p\| \cos \theta$$

where θ is angle between ∇f_p and \vec{u} .

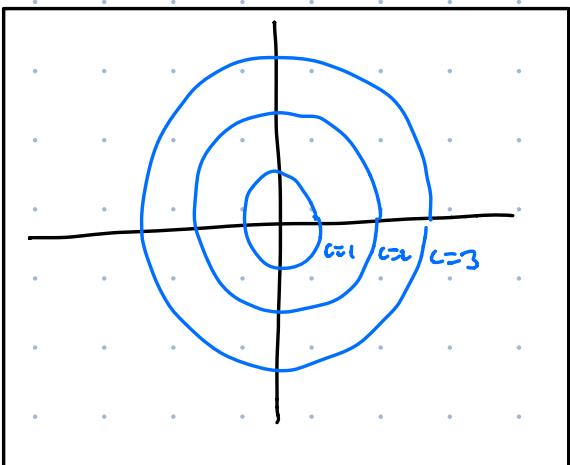
Problem 3 Calculate the directional derivative of $f(x, y) = e^{xy - y^2}$ at $P = (2, 2)$ in the direction of $\vec{v} = \langle 12, -5 \rangle$

Problem 4 Suppose $f(x, y)$ is a function such that at a point P , we have $\|\nabla f_p\| = 3$. What is the directional derivative in the direction $\pi/6$ radians away from the direction of fastest increase.

3d level surfaces and tangent planes.

Compare: from a couple of weeks ago
we have contour plots.

i.e. $f(x,y) = x^2 + y^2$

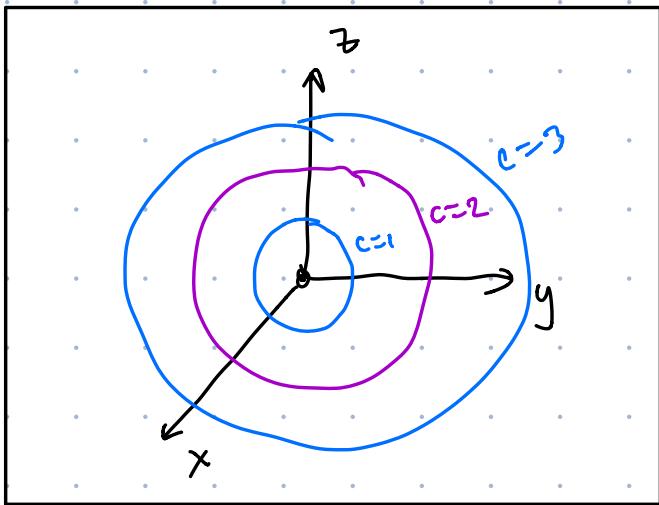


each curve signifies different constant values the function outputs for points on that curve.

we can do the same for 3-variable functions
and get level surfaces.

i.e. $f(x,y,z) = x^2 + y^2 + z^2$

Then $f(x,y,z) = 1$ gives sphere radius 1
 $f(x,y,z) = 2$ \longrightarrow $\sqrt{2}$
 $f(x,y,z) = 3$ \longrightarrow $\sqrt{3}$



pt3 on each sphere
into f output
the corresponding
c values.

- Going along the level surface, $f(x,y,z)$ doesn't change. Hence ∇f is \perp to the level surface.

We can use this to easily find normals to surfaces (and hence tangent planes).

Example

Find tangent plane to $xz + 2x^2y + y^2z^3 = 11$ at $P = (2, 1, 1)$.

- Write equation as level surface.

Let $f(x,y,z) = xz + 2x^2y + y^2z^3$. Then surface is the level surface $f(x,y,z) = 11$.

(2) Find gradient at point:

$$\nabla F = \langle z + 4xy, 2x^2 + 2yz^3, x + 3y^2z^2 \rangle$$

$$\nabla F(2,1,1) = \langle 9, 10, 5 \rangle$$

(3) Find tangent plane.

Given by $\nabla f_p \cdot (\vec{x} - \vec{OP}) = 0$

$$\langle 9, 10, 5 \rangle \cdot \langle x-2, y-1, z-1 \rangle = 0$$

$$9(x-2) + 10(y-1) + 5(z-1) = 0.$$