Learning outcomes

(1) Be able to calculate tangent planes

(2) Understand linear approximation as replacing functions with a tangent plane to get estimates

(3) Be able to use linear approximation to get estimates for simple functions.

(4) understand differentiable at a point as meaning error of function and linear approximation goes to zero faster than a linear function.

(5) Understand tangent plane as th plane of all tangent vectors of curves that go through a specific point.

Tangent planes

If $f(x,y)$ is differentiable at (a,b), then the tangent plane at (a,b) is given by:

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Remember:

- $f_x(a,b)$ is slope of graph in positive x-direction
- $f_y(a,b)$ ——————————— " — y-direction

Problem 1 Find tangent plane of $g(x,y) = e^{x/y}$ at the point $(2,1)$.

Problem 2

(a) When is the tangent plane horizontal?
(b) What is a normal vector for the tangent plane?
(c) Can the tangent plane be vertical?

Linear approximation

The basic idea of linear approximation is to replace a function with it's tangent plane (called the "linearization") and use this to estimate function values.

linear functions are super nice since changing one of the inputs induces a proportional change

in the outputs.

Example $f(x,y) = 6 + 3x + 4y$.

Then at $(1,1)$ we have: $f(1,1) = 13$.

- For every unit change in x, this causes a 3 unit change in output.

  ie $f(1,1) = 13$
  $f(2,1) = 16$
  $f(3,1) = 19$
  $f(0,1) = 10$
  etc..

- For every unit change in y, this causes a 4 unit change in output

  $f(1,1) = 13$ $\quad f(1,0) = 9$
  $f(1,2) = 17$ $\quad$ etc..
  $f(1,3) = 21$

- Moreover, this change is additive. i.e I can do the change in each variable and add the changes. ie increase x and y by 1 then output increases by $3+4=7$.

- Note, change in a variable is written as Δx or Δy. So this relationship can be written as:

$$\underbrace{\Delta f}_{\text{change in output}} = f_x(a,b)\underbrace{\Delta x}_{\text{change in } x} + f_y(a,b)\underbrace{\Delta y}_{\text{change in } y}$$

Problem 3 Suppose f is a linear function with $f(1,2)=4$ and $f_x(1,2)=3$, $f_y(1,2)=-2$. Find:

(a) $f(2,2)$

(b) $f(2,4)$

(c) $f(4,1)-f(2,1)$

The linearization of a function at a point is just the tangent line.

ie, linearization at (a,b) is:

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Example: Look at e^{-xy} in desmos.
Tangent plane at $(2,2)$. Error for different pts.

Problem 9 use linear approximation of $f(x,y) = \sqrt{\frac{x}{y}}$ at $(9,4)$ to estimate $\sqrt{9.1/3.9}$

That is

- Find linearization (tangent plane) at $(9,4)$
- Treat f as if it was this linearization to estimate $\sqrt{9.1/3.9}$.

Differentiability

- Linearization can also be calculated if the function isn't differentiable (sometimes). In fact differentiability is defined by linearization.

A function $f(x,y)$ is differentiable at a point (a,b) if

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - L(x,y)}{\|\langle x-a, y-b\rangle\|} = 0$$

where $L(x,y)$ is the linearization of f.

Let's break this down:

- $f(x,y) - L(x,y)$ is the error of the linear

approximation at (x,y)

If $f(x,y)$ is continuous then $\lim_{(x,y)\to(a,b)} f(x,y) - L(x,y) = 0$

- $\|\langle x-a, y-b\rangle\|$ is the distance of (x,y) to (a,b).

  We have $\lim_{(x,y)\to(a,b)} \|\langle x-a, y-b\rangle\| = 0$.

- We can think of $\lim_{(x,y)\to(a,b)} \dfrac{f(x,y) - L(x,y)}{\|\langle x-a, y-b\rangle\|} = 0$

as saying the linear error $f(x,y) - L(x,y)$ goes to zero faster than the distance from $\langle a,b\rangle$ to $\langle x,y\rangle$ goes to zero.

ie $L(x,y)$ is a "very good" approximation.

Problem 5 True or false

(1) If the partials $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ exist, then the function is differentiable at a point

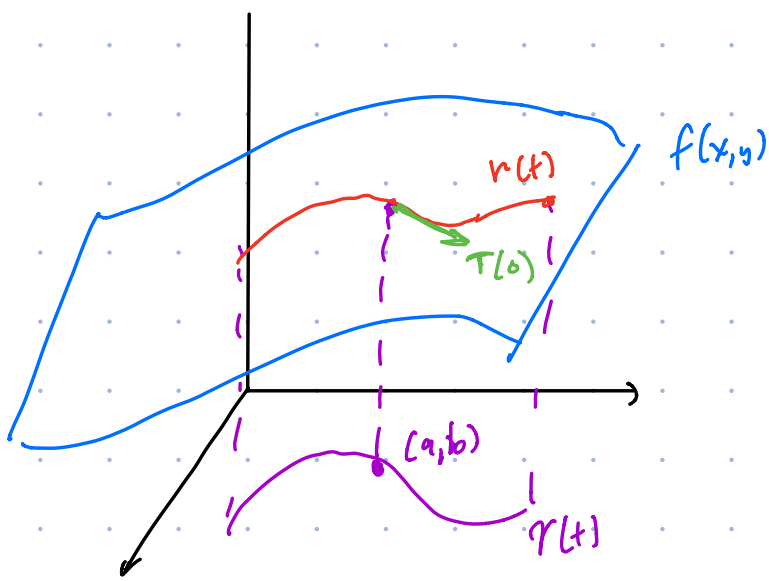
(2) If a function is differentiable at a point,

then it is continuous at that point.

Alternate description of tangent plane

Given any curve parametrization $r(t) = \langle g_1(t), g_2(t) \rangle$ such that $r(0) = \langle g_1(0), g_2(0) \rangle = \langle a, b \rangle$. Then if $f(x,y)$ is differentiable at (a,b), then consider the curve in 3-space given by

$$r(t) = \langle g_1(t), g_2(t), f(g_1(t), g_2(t)) \rangle$$



we than have a tangent $\vec{T}(0)$ and this is contained inside the tangent plane at (a,b). Infact for any vector contained inside the tangent plane, we can find a curve $r(t)$ such that the tangent $\vec{T}(0)$ to the corresponding $r(t)$ is that vector.

- You can use this to show something isn't differentiable at a point. ie
  - have a candidate tangent plane from the partial derivatives
  - show that for some curve $\gamma(t) = \langle \gamma_1(t), \gamma_2(t) \rangle$ that the tangent vector $\vec{T}(0)$ of the curve $\vec{r}(t) = \langle \gamma_1(t), \gamma_2(t), f(\gamma_1(t), \gamma_2(t)) \rangle$ isn't in the tangent plane.