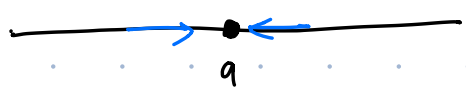
Limits and continuity

Informally:

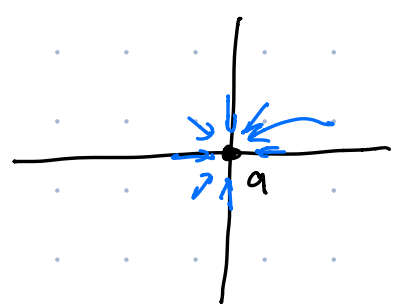
A limit of a function at some input $x=a$ is some value L (the limit) such that no matter what you input into the function, if the inputs get closer and closer to a, (but not equal) then the outputs get closer and closer to L.

- This idea is the same for one variable functions and multivariable functions.

- The difficulty comes from the fact in 1-variable, you can approach an input either from the left and right only. In multiple variables there are infinitely many ways to approach something.



one-variable.



2-variables.

- You usually prove multivariable limits dont exist by showing that the limit along two different paths to the value are different.

Example $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ doesn't exist.

Why? Along the x-axis, $y=0$. Hence,

$$\lim_{(x,0)\to(0,0)} \frac{x^2-0}{x^2+0} = \lim_{(x,0)\to(0,0)} 1 = 1$$

Along y-axis, $x=0$: $\lim_{(0,y)\to(0,0)} \frac{0-y^2}{0+y^2} = -1.$

These are different, so the limit doesn't exist.

Problem 1: Does $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+2y^2}$ exist?

Hint: A line isn't good enough.

Showing a limit exists is either easy or very hard.

- Simple case: the function is continuous. Then you can substitute the value into the function

(continuity itself means that the limit at a point is equal to the value of the function of that point. That is $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.)

Warning To show a limit exists, it is not enough to show the limit along all lines or some paths all converge to the same thing.

To show a limit exists without continuity usually means using the squeeze theorem in some way.

setup for sequeeze: $\lim_{(x,y)\to(a,b)} f(x,y)g(x,y)$

where one function say $g(x,y)$ is wild but bounded, and $f(x,y)$ is nice with known limit. Then we take:

$$m \le g(x,y) \le M$$

$$mf(x,y) \le f(x,y)g(x,y) \le Mf(x,y) \qquad (\text{if } f>0)$$

take limit on both sides, if both are equal then the middle limit is also this.

Example $\lim_{(x,y)\to(0,0)} (x^2+y^2)\cos\left(\frac{1}{x^2+y^2}\right)$

- Wild but bonded function: $\cos\left(\frac{1}{x^2+y^2}\right)$
- nice function: x^2+y^2

$$-1 \le \cos\left(\frac{1}{x^2+y^2}\right) \le 1$$

$$-(x^2+y^2) \le (x^2+y^2)\cos\left(\frac{1}{x^2+y^2}\right) \le x^2+y^2$$

The left and right functions are continuous, so

$$\lim_{(x,y)\to(0,0)} -(x^2+y^2) = 0 = \lim_{(x,y)\to(0,0)} x^2+y^2.$$

Hence $\lim (x^2+y^2)\cos\left(\frac{1}{x^2+y^2}\right) = 0.$ ∎

Problem 2 Prove $\lim_{(x,y)\to(0,0)} \tan^2 x \sin\left(\frac{1}{|x|+|y|}\right) = 0.$

Partial derivatives.

Informally: The partial derivatives of a function $f(x,y)$ at a point $P=(a,b)$ can be viewed as:

- $\left.\frac{\partial f}{\partial x}\right|_P$ = the slope of f in the direction of positive x-axis

- $\left.\frac{\partial f}{\partial y}\right|_P$ = ——— " ——— ——— positive y-axis.

Example:

Demonstration with Desmos app with $f(x,y) = \cos(x+y)$.

- Calculating partial derivatives is as simple as considering the other variables as constant.

Problem 3 Find the partial derivatives of $f(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}$.

- we can differentiate again to get higher order partials.

Example: Consider $f(x,y) = 6x^2 + 2xy + 3y^2$

$$\frac{\partial f}{\partial x} = 12x + 2y \quad , \quad \text{and} \quad \frac{\partial f}{\partial y} = 2x + 6y$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 12$$

ie $\dfrac{\partial^2 f}{\partial x^2} = 12$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(12x + 2y\right)$$

$$\frac{\partial^2 f}{\partial y \partial x} = 2$$

etc...

Clairut's Theorem: under nice conditions, $f_{xy} = f_{yx}$. ie it doesn't matter what order you differentiate.

Extra things to talk about.

- absolute value in squeeze
- last problem in H/W.