

Problem 1

$\int_0^T \vec{r}'(u) du$ is the total displacement after T seconds.

This can be seen as a consequence of the fundamental theorem of calculus. In particular, suppose

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

$$\text{Then } \vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

$$\text{Hence, } \int_0^T \vec{r}'(u) du = \left\langle \int_0^T x'(u) du, \int_0^T y'(u) du, \int_0^T z'(u) du \right\rangle$$

$$= \langle x(T) - x(0), y(T) - y(0), z(T) - z(0) \rangle$$

$$= \vec{r}(T) - \vec{r}(0)$$

Hence, as the bee starts at the origin and $\int_0^T \vec{r}'(u) du = \vec{r}(T) - \vec{r}(0)$.
This means the bee is at the origin at time T .

$\int_0^T \|\vec{r}'(u)\| du$ represents the total distance the bee has travelled in T seconds.

Problem 2

(a)

$$\vec{r}(t) = \langle e^t \sin t, e^t \cos t, e^t \rangle.$$

$$\vec{r}'(t) = \langle e^t \sin t + e^t \cos t, e^t \cos t - e^t \sin t, e^t \rangle$$

$$\begin{aligned}\|\vec{r}'(t)\| &= \sqrt{e^{2t}(\sin t + \cos t)^2 + e^{2t}(\cos t - \sin t)^2 + e^{2t}} \\ &= e^t \sqrt{(\sin t + \cos t)^2 + (\cos t - \sin t)^2 + 1} \\ &= \sqrt{3} e^t\end{aligned}$$

(b)

Arclength s as a function of time t :

$$s(t) = \int_0^t \|\vec{r}'(u)\| du = \sqrt{3} \int_0^t e^u du = \sqrt{3} (e^t - 1)$$

(c)

time t as a function of arclength s .

$$s = \sqrt{3} (e^t - 1)$$

$$s/\sqrt{3} = e^t - 1$$

$$e^t = \frac{s}{\sqrt{3}} + 1$$

$$t = \ln \left(\frac{s}{\sqrt{3}} + 1 \right)$$

(d)

The arclength parameterization is then given

by:

$$\vec{r}_1(s) = \vec{r}(t) = \vec{r}\left(\ln\left(\frac{s}{\sqrt{3}} + 1\right)\right)$$

$$= \left\langle \left(\frac{s}{\sqrt{3}} + 1\right) \sin\left(\ln\left(\frac{s}{\sqrt{3}} + 1\right)\right), \left(\frac{s}{\sqrt{3}} + 1\right) \cos\left(\ln\left(\frac{s}{\sqrt{3}} + 1\right)\right), \left(\frac{s}{\sqrt{3}} + 1\right) \right\rangle$$

Problem 3

- (a) The curvature of a circle of radius R is $K = 1/R$. Since $1/3 > 1/4$ this is true.

- (b) False. The definition of curvature is given by

$$K(s) = \left\| \frac{d\vec{T}}{ds} \right\|$$

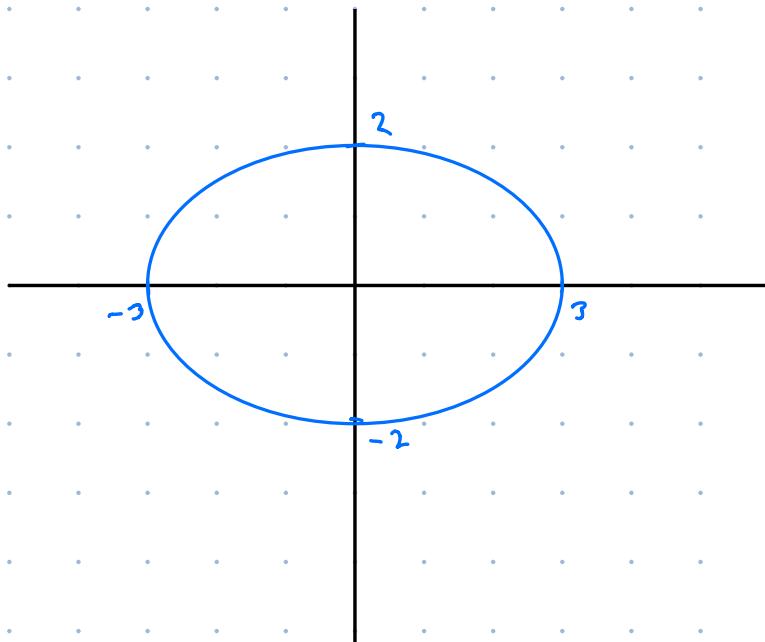
which is in terms of an arclength parameterisation. Arclength parameterisations only differ by constants and so don't change the derivatives.

- (c) False. Curvature only gives how much the curve is bending and not the direction it is bending towards.

Something like the helix can also have constant curvature.

Problem 4

(a)



minimal at $(0,2), (0,-2)$, maximal at $(3,0), (-3,0)$
(consider osculating circles at those points)

(b) An ellipse is parameterised in 3-space by:

$$\vec{r}(t) = \langle 3\cos t, 2\sin t, 0 \rangle$$

$$\vec{r}'(t) = \langle -3\sin t, 2\cos t, 0 \rangle$$

$$\vec{r}''(t) = \langle -3\cos t, -2\sin t, 0 \rangle$$

$$\vec{r}' \times \vec{r}'' = \langle 0, 0, 6\sin^2 t + 6\cos^2 t \rangle$$

$$= \langle 0, 0, 6 \rangle$$

$$\therefore \|\vec{r}' \times \vec{r}''\| = 6$$

$$\|\vec{r}'\|^3 = (9\sin^2 t + 4\cos^2 t)^{3/2}$$

Hence $k(t) = \frac{6}{(9\sin^2 t + 4\cos^2 t)^{3/2}}$

(C) The curvature from the previous part is minimal/maximal when $9\sin^2 t + 9\cos^2 t$ is maximal/minimal respectively.

Hence, minima occur when $t = \frac{\pi}{2} \pm n\pi$, maxima when $t = n\pi$, $n \in \mathbb{Z}$.

so minima at the points $\vec{r}\left(\frac{\pi}{2} \pm n\pi\right) = \langle 0, \pm 2 \rangle$

maxima at the points $\vec{r}(n\pi) = \langle \pm 3, 0 \rangle$.